INSIGHT. Revisions and Excisions.

1. All references to MS are to page numbers written in ink in upper right hand corner.

2. Remove previous pro~~x~~face [MS pages 1 - 8] and insert new preface [pages i - xiii].

3. Introduction. Delete last two lines [MS page 24] and remove MS page 25. Insert new page 25.

4. Chapter I. Delete last two lines [MS page 58] and remove MS pages 59 - 75. Insert new pages 59 - 75.

5. Chapter II. Remove MS pages 85 - 107 and insert new pages 85 - 107.

6. Chapter III. Remove MS pages 150 - 159 and insert new pages 150 - 152.

7. Chapter V. Remove MS pages 261 - 265 and insert new pages 261-264.

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N. B. There are nineteen additional pages in Chapter II but sixteen fewer pages in Chapters III and V.

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INSIGHT

PREFACE

Rational self-consciousness is a peak above the clouds. Intelligent and reasonable, responsible and free, scientific and metaphysical, it stands above romantic spontaneity and the psychological depths, historical determinism and social engineering, the disconcerted existential subject and the undeciphered symbols of the artist and the modernist.

Yet if man can scale the summit of his inner being, also he can fail to advert to the possibility of the ascent or, again, he can begin the climb only to lose his way. If then he knows himself as in fact he is, he can know no more than that he has been cast into the world to be afflicted with questions he does not answer and with aspirations he does not fulfill. For it is the paradox of man that what he is by nature is so much less than what he can become; and it is the tragedy of man that the truth, which portrays him as actually he is, can descend like an iron curtain to frustrate what he would and might be.

Facts, it is said, are stubborn things. But there is a sense in which, I believe, it is true to say that the facts about man can be out-flanked. For a change in man, a development of potentialities that are no less real because, like all potentialities they are latent, not only is itself a fact but also can be a permanent source of new facts that cumulatively alter the complexion of the old.

So it is that the present work is a program rather than an argument. It begins not by assuming premises but by presuming readers. It advances not by deducing conclusions from the truths of a religious faith or from the principles of a philosophy but by issuing to readers an invitation, ever more precise and more detailed, to apprehend, to appropriate, to envisage in all its consequences, the inner focus of their own intelligence and reasonableness. That focus, it will be claimed, is insight. But to apprehend the focus is to gain insight into insight, to pierce the outer verbal and conceptual exhibitions of mathematics, of science, and of common sense, and to penetrate to the inner dynamism of intelligent inquiry and critical reflection. To appropriate the focus is both to know and to know what it is to know one's own intelligence, one's own reasonableness, one's own essential and restrictedly effective freedom. To envisage the focus in the full range of its implications is to discover for oneself what is meant by being, by objectivity, by metaphysics, by ethics, by God, and by evil.

Frankly, even as a program, even as a sketch that offers only to indicate the detailed map that is needed, the present work may be reproached for excessive ambition. But if I may borrow a phrase from Ortega y Gasset, one has to strive to mount to the level of one's time. The twentieth century has been described as the end of the Renaissance. Some four centuries ago there was projected a new world: new nations had arisen in new political constellations; a new art was matched with the promise of a new science; and new philosophies were disseminated through a new education. That new world has been realized, but the ideas that fostered its genesis have been discredited by its maturity. What was so new has become so old. To have been educated is no longer a matter of speaking Latin and writing Greek. Modern art would puzzle Raffaello, as modern technology would astound da Vinci. The new nations are not in Europe, and the issues of modern politics seem transcribed from the pages of Utopia. Einstein has revised momentously the thought of Galileo, and Heisenberg has contended that good Laplace, like Homer, nods. The novel outlook that is transforming the natural sciences cannot but affect profoundly the methods that were transferred with so sedulous a fidelity from the natural to the human sciences. Not even Renaissance ridicule of the Middle Ages has been able to prevent a rebirth of interest in logic. Not even the Enlightenment's insistence on the autonomy of man has been able to prevent the recurrence of theological themes under the guise of existentialist philosophy.

So it is that a new world has been bequeathed us and yet we, the heirs of the Renaissance, have been denied its spirit of bold confidence, of venturous assurance. For we know too much in too many fields, we have witnessed too much suffering in too many unexpected quarters, to purchase confidence by an easy exuberance of feeling or to accept words of assurance without answers to our questions. Nor was the basic question missed, when the late Prof. Ernst Cassirer, towards the end of a long and highly productive career, endeavored to communicate within a brief compass some of the main conclusions of his vast erudition and ever penetrating thought. Just what is man? Answers, he remarked, have been worked out by theologians and scientists, by politicians and sociologists, by biologists and psychologists, by ethnologists and economists. But not only do the many answers not agree, not only is there lacking some generally accepted principle that would select one and reject the others, but even within specialized fields there seems to be no method that can confront basic issues without succumbing to individual temperament and personal evaluations.

In the midst of this widespread disorientation, man's problem of self-knowledge ceases to be simply the individual concern inculcated by the ancient sage. It takes on the dimensions of a social crisis. It can be read as the historical issue of the twentieth century. If in that balance human intelligence and reasonableness, human responsibility and freedom, are to prevail, then they must be summoned from the dim and confused realm of latent factors and they must burst forth in the full power of self-awareness and self-possession.

If such is the urgency of personal appropriation of rational self-consciousness, the difficulty of achievement should not discourage attempts at making a beginning. If the extent and the complexity of modern knowledge preclude the possibility in our time both of the uomo universale of the Renaissance and of the medieval writer of a Summa, at least the collaboration of many contains a promise of success, where the unaided individual would have to despair.

Still a collaboration has its conditions. It supposes a common vision of a common goal. It supposes at least a tentative idea that would unify and coordinate separate efforts in different fields. It supposes a central nucleus that somehow could retain its identity yet undergo all the modifications and enrichments that could be poured into its capacious frame from specialized investigations.

It is with the conditions, preliminary to an effective collaboration, that the present work is concerned.

For in the measure that potential collaborators move towards a personal appropriation of their rational self-consciousness, in the same measure they will begin to attain the needed common vision of the common goal. In the measure that they discover in themselves the structure of developing intelligence, in the same measure they will share a tentative idea that can unify and coordinate separate efforts in different fields. In the measure that they reach the invariants of intellectual development, in the same measure they will possess a central nucleus that retains its identity through all the possible developments of human intelligence.

Prof. Cassirer has told us that, from the viewpoint of a phenomenology of human culture, the explanatory definition of man is <u>animal symbolicum</u> rather than <u>animal rationale</u>. But in the measure that men appropriate their rational self-consciousness, not only do they re-establish the <u>animal rationale</u> but also they break through the phenomenological veil. For, as will be argued, they can reach a universal viewpoint from which individual temperament can be discounted, personal evaluations can be criticized, and the many and disparate reports on man, emanating from experts in various fields, can be welded into a single view.

But if I believe that man's self-awareness and self-possession can add a further, overarching component to Prof. Cassirer's portrayal of man, it is not to be overlooked that a possibility is claimed and not an achievement. I could not convey my meaning without venturing into many fields, into mathematics and physics, into the subtleties of common sense and depth psychology, into the processes of history, the intricacies of interpretation, the dialectic of the philosophies, and the possibility of transcendent knowledge. I would not wish anyone to entertain the fanciful nonsense that I can speak with authority or even competence in so many fields. I do not expect many experts to recognize their science in the formulations that suit my purpose. Yet, perhaps, I may hope that there will be some that share my preoccupations and interests, that will divine what I am endeavoring to say and will proceed to say it more adequately, that will grasp how my ignorance and oversights can be remedied without completely invalidating the fundamental structures that make possible a common vision of a common goal. Finally, if in any measure that hope is realized, ~~fulfilled~~, the relative isolation of my efforts will have ended and the preliminary conditions will begin to be fulfilled for the collaboration I would merely initiate.

It is customary to conclude a preface with an acknowledgement of one's indebtedness. Naturally I am inclined to think in the first place of the teachers and writers who have left their impress upon me in the course of the twenty-seven years since first I was initiated into epistemological issues. But so long a gestation contains too many half-lights, too many detours, for me to indicate in a brief yet intelligible fashion my proximate sources. So it is that I must be content to restrict my expression of gratitude to immediate benefactors: to the staff of L'Immaculée Conception, Montreal, where the underlying studies on Gratia Operans[1] and Verbum[2] were undertaken; to the staff of the Jesuit Seminary, Toronto, where I enjoyed the freedom to write the present work; to the Rev. Joseph Wulftange, Rev. Joseph Clark, Rev. Norris Clarke, Rev. Frederick Crowe, Rev. Frederick Copleston, and Rev. Andrew Godin, who ~~were~~ generously read the typescript and gave me the benefit of reactions and criticisms from specialists in different fields; to the Rev. Patrick Plunkett who labored (to my shame, rather vainly) to reduce the solecisms of my style; and to the Rev. Eric ~~O'Connell~~ O'Connor who was ever ready to allow me to draw upon his knowledge of mathematics and of science.

June 1949 to September 1953.

1) St. Thomas' Thought on Gratia Operans, Theological Studies [Woodstock, Md.] II(1941), 289-324; III(1942), 69-88, 375-402, 533-578.

2) The Concept of Verbum in the Writings of St. Thomas Aquinas, Ibid. VII(1946), 349-392; VIII(1947), 35-79, 404-444; X(1949), 3-40, 359-393.

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~~more accurate and more comprehensive insights at their own dis-~~ From a logical viewpoint it might seem that enough
has been
~~poss~~ been said. For the present book is not an argument but a program. It begins not by assuming premises but by presuming readers. It advances not by logical necessity but by the probabilities of vital growth. It aims not to determine objects by exact definitions, correct statements, and rigorous inferences, but to develop subjects by the prior communication of insights that always must be invoked if definitions are to be endowed with an accessible meaning, statements with an objective reference, inferences with a real significance. Its goal is an insight into insight and, since that goal can be reached from any sufficiently diversified set of insights, specialist readers always can and sometimes should replace my elementary illustrations by the more accurate and more comprehensive insights at their own disposal. They always can do so, for an illustration of insight is elementary because it omits corrections, qualifications, refinements; but all such omitted complexity is the fruit of further complementary insights; and specialist readers always can find these further insights by asking why they find the elementary account unsatisfactory. Moreover, they sometimes should do so. For while the further insights add nothing to the illustration of direct or inverse insight and become relevant to our purpose only when we come to treat the reflective insight in Chapter X, still for specialist readers to neglect them is not only to miss a very favorable opportunity to grasp what insight is and how it develops but also to risk encouraging an oversight of insight and even a flight from understanding.

~~To turn from logical to metaphysical considera-~~

To turn from logical to metaphysical consideraa sequence of lower contexts for the purpose of reaching an upper context; and the basic upper context is to be pre-logical, not in the sense made current by M. Lévy-Bruhl, but in the sense that developing intelligence and reasonableness are prior to intelligently grasped and reasonably affirmed utterances. Still it may not be amiss to indicate a single instance in which the genetic order of developing insights differs from the logical order of defining thought. Thus, logically it is illegitimate to speak, for example, of the equality of the spokes of a cart-wheel without explaining that the spokes will be said to be equal if the same number is reached in measuring each of them. In turn, this statement calls for a further statement in which the meaning of the word, measuring, is explained; and that explanation calls for an account of units of measurement, of their standardization, of the numbers employed in measuring, and of the isomorphism of mathematical and physical relations. On the other hand, genetically it seems clear enough that Euclidean geometry existed for some centuries before there occurred any effective advertence to its metrical suppositions. More generally, it seems true that prior to every correct logical formalization there is a sufficiently univocal communication of insights, that this prior communication grounds not only non-technical discourse but also the possibility of discussing the adequacy or inadequacy of any formalization, and that from a pedagogical viewpoint the correct procedure is to begin by communicating the insights.

To turn from logical to metaphysical considera-

To turn from logical to metaphysical considerations, I had best begin by explaining a probable paradox. For I think it likely that I may succeed in persuading some positivists of the existence and validity of metaphysics yet, at the same time, draw from hitherto convinced metaphysicians outraged protests that I have turned everything upside down and inside out. To elucidate such a bewildering phenomenon, it will be helpful to recall Aristotle's distinction between the <u>causa essendi</u> (the moon goes through these phases, because it is a sphere) and the <u>causa cognoscendi</u> (the moon is a sphere, because it goes through these phases). Now the traditional presentation of metaphysics has been in terms of the <u>causa essendi</u>, and only incidentally has the inverse relationship of the <u>causa cognoscendi</u> received attention. But in the present work the whole conception of metaphysics is dominated by the <u>causa cognoscendi</u>; and while the labor of writing a supplementary volume would reveal in detail the equivalence of the two presentations, still so great a labor would be superfluous for anyone willing to attend to a rather brief argument.

Let us suppose that metaphysics is a science and that a science is <u>certa rerum cognitio per causas</u>. Let us also suppose that there exists an ontologically structured metaphysics, that is, that there is a department of certain knowledge of things in which, in all basic instances, the ground or cause is a <u>causa essendi</u>. Then either it is or it is not possible to explain how the ontologically structured metaphysics is known. If that explanation is possible, then the ontologically structured metaphysics in its entirety is deducible from a ground or set of grounds in which regularly the causa cognoscendi is assigned. So on this first alternative the ontologically structured metaphysics necessarily supposes an epistemologically structured metaphysics. But on the second alternative, on which one cannot know how the ontologically structured metaphysics is known, there arises a series of disconcerting questions. For if one cannot know how it is known, then there can be no method of metaphysics: it can be claimed that results are obtained; but it cannot be suggested that anyone can know how to go about obtaining them. Again, if one cannot know how the ontologically structured metaphysics is known, there arises the suspicion that it is not known but merely asserted; for if one cannot know how knowledge is acquired, how can one know that it is possessed. Again, as will appear, it is possible to explain just how mathematics is known, just how natural science is known, just how common sense is developed, just how beliefs are acquired and spread. It follows that the unknowable genesis of ontologically structured metaphysics can have nothing in common with the genesis of mathematics, nothing in common with the genesis of natural science, nothing in common with the genesis of common sense, and nothing in common with the genesis of belief. Indeed, even though mystical knowledge and extra-sensory perception have not been explained, no one claims that they cannot ever be explained; and so it would follow that the unknowable genesis of ontologically structured metaphysics is even more ~~un-~~ inscrutable than mystical experience and extra-sensory perception.

In brief, while there is a certain novelty to my approach to metaphysics, the novelty lies not in the principle but in the achievement. The notion of the _causa cognoscendi_ is as old as the notion of the _causa essendi_. If an ontologically structured metaphysics can be known, then the activity of knowing it can be known. If that activity can be known, then the activity supplies a premise from which the known can be deduced. Though the deduction is new, still it can be rejected in principle only by affirming in principle that metaphysics cannot be methodical, cannot but be suspect, cannot but be classed as more mysterious than mysticism and extra-sensory perception. Such consequences are no more acceptable to the metaphysicians of the present than to those of the past; and so one is driven to accepting the first alternative, namely, that one cannot affirm an ontologically structured metaphysics without supposing (I do not say "without knowing") an epistemologically structured metaphysics in which everything may seem, especially to the routine mind, to be inside out and upside down because the argument runs not from the _causa essendi_ but from the _causa cognoscendi_.

Our aim, then, regards 1) not the fact of knowledge but a discrimination between two facts of knowledge, 2) not the details of the known but the structure of the knowing, 3) not the knowing as an object characterized by catalogues of abstract properties but the appropriation of one's own intellectual and rational self-consciousness, 4) not a sudden leap to appropriation but a slow and painstaking development, and 5) not a development indicated by appealing either to the logic of the as yet unknown goal or of pure gold without melting it down? What accounts for a wheel being round? What is arithmetic and how does one go on to algebra? In each case, there is an appropriate image or set of images that, under the stress of inquiry, results in an insight that expresses itself in some formulation called the answer.

Now attention has to be directed to a quite different case. There is the question. There is the answer. But the answer consists in showing the question to be misconceived, and it is grounded in an insight that grasps why the question, as conceived, cannot be answered.

4.1 Surds. How big is the square root of two? Clearly, it is greater than one, for the square ~~root~~ of one is one; and it is less than two, for the square of two is four. It would seem, then, that it is some improper fraction lying between one and two.

Now an improper fraction is the quotient of some positive integer divided by some other, smaller positive integer. Moreover, it is always possible to reduce such a fraction to its lowest terms by removing all common factors. Let us suppose then, that:

$$\sqrt{2} = M/N$$

where M and N are positive integers with no common factors. Multiplying across by N and squaring, one obtains:

$$2N^2 = M^2$$

It follows that M must be an even number and so twice, say, P.

Substituting and dividing by two, one obtains.

$$N^2 = 2P^2$$

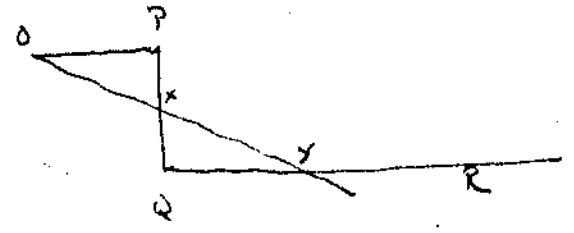
so that N also must be an even number, which contradicts the assumption that all common factors were eliminated. It follows that there is no "rational" fraction, M/N, that is equal to the square root of two. Moreover, since any recurring decimal can be reduced to such a fraction, there is no recurring decimal equal to the square root of two. However, one can apply to 2 the ordinary method for taking the square root, and so it remains that the square root of two will be an infinite, non-recurring decimal. Finally, the foregoing argument can be generalized and applied to any surd. Thus, if

$$3N^2 = M^2$$

then 3 must be a factor of M, so that M can be replaced by 3P, whence, it will follow that 3 must be a factor of N.

Non-countable Infinity.

4.2 Again, to raise another, similar question. How many points are there on a straight line one inch long? Clearly, the number must be very large, for a point is position without magnitude. But, at least, one would be inclined to say that there are twice as many points in a straight line two inches long as in a straight line one inch long. Still, that would be erroneous, as appears from the following construction. Let the straight line, PQ, be perpendicular to the straight lines, OP and QR. Let QR be twice as long as PQ. And let OXY be a straight line cutting PQ in X and QR in Y.



Then, from the construction, it is clear that for every point, Y, in QR, there is a corresponding, distinct point, X, in PQ. Indeed, this remains true if one produces QR to infinity in the direction of R. No matter where Y is taken on QR produced, there is always a corresponding and distinct point X in PQ. Hence, there are as many points in an inch of straight line as there are in two inches, or in a foot, or in a mile, or in as many light-years as you please.

However, we have not met the question. We have said there are as many as ..... We have not said how many. Accordingly, let us distinguish between the counted, the countable, and the non-countable. A set is counted when one says it contains N members, where N is some positive integer. A set is countable when it can be arranged in some determinate order that contains all its members once each and only once; for then there can be established a one-to-one correspondence between the members of the set and the positive integers. Finally, a set is non-countable when it is not possible to establish a one-to-one correspondence between its members and the positive integers.

It is to be noted that by "countable" is not meant the possibility of finishing the counting. Thus, an infinite series, such as

1/2, 1/4, 1/8, 1/16, ...

is countable, for all its members lie in a determinate order and so can be placed in a one-to-one correspondence with the positive integers. Again, an infinite series of infinite series of elements is countable, for all its elements can be regarded as lying within a single determinate order. Thus, the reciprocals of the <u>n</u>th powers of the prime numbers form an infinite series of infinite series. Their elements can be arranged in a column of rows, thus:

1/2	1/4	1/8	1/16	....
1/3	1/9	1/27	1/81	....
1/5	1/25	1/125	1/525	....
1/7	1/49	1/343	1/2401	....

and any column of rows can be counted in the following manner:

1	2	5	10	17
4	3	6	11	etc....
9	8	7	12	etc....
16	15	14	13	etc....

Thus, any infinite series of infinite series can be assigned the order of a single infinite series. It follows that an infinite series of infinite series of infinite series can be arranged in a column of rows and so can be assigned the order of a single infinite series. The theorem can be repeated indefinitely. Thus, consider the rational, proper fractions:

1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 1/6, ...

From this infinite series there can be derived an infinite series of infinite series, for one can take, first, the square root of the lot, then, the cube root, then square the cube roots, then take the fourth root, then cube the fourth roots, etc...Now, as has been shown, this infinite series of infinite series can be arranged in a single series. Once this is done, one can use these new terms as powers to be applied to the rational proper fractions to derive a new infinite series of infinite series. This can be arranged in a single series, applied as powers to the rational proper fractions, yield a new infinite series of infinite series, etc., etc.

From the foregoing it is clear that any infinite set is countable, provided it is possible to assign some order to its members. It is also clear that a non-countable infinite set must contain such a multitude of members in such a manner that ordering them is impossible. Such is the case with the points in a straight line. Thus, in the line, QR, it is impossible to pick any point, Q', that is nearest to Q: for however short QQ' may be, it contains as many points as there are in a line as long as you please. Nor is there any use trying to proceed by dividing the line. For if this could be done in an orderly fashion, then one would be appealing to an ordered series of all the numbers greater than zero and less than unity. But the range of numbers is a non-countable infinite set, for it cannot be arranged in a single order. Suppose there were some single column containing all the infinite decimals. Then consider the diagonal. It is always possible to construct another infinite decimal that differs from the first infinite decimal by the digit in the first place, from the second by the digit in the second place, from the nth by the digit in the nth place. Therefore, the initial assumption is false. The column did not contain all the infinite decimals. There is, then, no single series that contains all the infinite decimals and so the infinite decimals are a non-countable infinite set.

Well, how many points are there in a straight line an inch long? There is no answer. They form a non-countable infinite set. They do so, because they cannot be placed in a single order and so cannot be correlated in a one-to-one correspondence with the positive integers. However, they can be placed in a one-to-one correspondence with other non-countable infinite sets. Thus, there are as many points in an inch as in a mile or in a light-year or in as many light-years as you please. But that does not mean that there is some determinate number of points in an inch or in a mile. Much less does it mean that some smaller number is equal to a greater number. There just is no numbering, no counting. And there is no numbering or counting because there is no possibility of effecting an order, a system, an arrangement.

4.3 Function and Limit.

One might think that this exclusion of number and of order blocked the mathematician. In fact, it gives him a new lease of life. What is the mathematician's continuous function? In the elementary case, it is a one-to-one correspondence between non-countable infinite sets. Moreover, since such a correspondence can be set up between an inch and a foot, or an inch and a mile, or an inch and a light-year, or any intermediate or still odder pair, since, visually, length is independent of the number of points, the mathematician can develop the infinitesimal calculus. But he does so, not by finding some order in the non-countable infinite set, but by developing a technique of getting around it. This technique is named proceeding to the limit.

Thus, consider the continuous function, $y = x^2$. It is a function if, for every value of x, there is a corresponding value of y. It is a continuous function, if the values of x are a non-countable infinite set, ~~and for every distinct value of x there is a distinct value of y.~~

Now as x increases, y must increase more rapidly, for it equals the square of x. Hence, visually, as one moves from point to point along x, one must move more rapidly from point to point along y. Moreover, the further one advances along x, the greater must be one's strides along y. Still, there are no points omitted along x and there are no points omitted along y.

What, then, is the ratio of the increment of y to the increment of x ? Clearly, if x increases by some slight amount, h, y will increase by

$$(x + h)^2 - x^2 = 2xh + h^2$$

Hence the ratio of the corresponding increment of y to the increment, h, of x will be $(2x + h)$. The smaller the increment, h, the nearer is the ratio to 2x. In the limit, it is exactly 2x. Thus, if the limit of the ratio of the increment in y to the increment in x is denoted by the symbol, dy/dx, then besides the initial function, $y = x^2$, we have also the derivative function, $dy/dx = 2x$.

Now what is this business of proceeding to the limit? There is said to be a limit, P, to a non-determined quantity, Q, if the difference of Q from P can be made smaller than any number one cares to assign. Thus, above, by making the increment, h, smaller and smaller, one can make the difference of (2x + h) from 2x as small as we please. Still, this is only the conceptual formulation of the procedure of taking a limit. What is the underlying insight? What is the image that the insight presupposes?

Clearly enough the image will differ in different cases. Similarly, the insight will be reached in different manners. But the peculiarity of the insight is that it grasps, not that something is to some point, but that something is beside the point. No matter how small h is, there is a non-countable infinite set of values between 2x and (2x + h). They are non-countable because they defy arrangement, order, system. They exhibit a non-systematic aspect of continuous variables and continuous functions. But what one is trying to do in mathematics, is to reach the systematic. If that is all one wants, one can disregard the non-systematic. One can leap over the non-countable infinity because it is without order if one's aim is to grasp just what admits order. Again, the ratio of the increment of y to the increment of x is any of a non-countable infinite set of values. But the limit of that ratio is unique. It can be determined systematically. It pertains to system.

Abstraction.

4.4/ When one comes to think of it, we have been doing this sort of thing all along. The principles of displacement and of specific gravity would not enable Archimedes to determine that there was nothing but pure gold in the crown; they would enable him to say merely that there was extremely little else. Again, the definition of the circle paid no attention to the size, the weight, the strength, the origin, the materials, the purpose of the cart-wheel; on the contrary, it went off to a realm of the non-imaginable where points have position without magnitude and lines have length without thickness. Finally, the transition from arithmetic to algebra did not consist in paying closer attention to the things one might count by the positive integers; it consisted in deserting the good, common sense notion of adding and in developing a new notion that gave a meaning to adding negative numbers, multiplying fractions, and doing other things that have no prima facie meaning.

It is time, then, for us to reflect on certain general aspects of the process from image through insight to conceptions, and so we had best begin a new section.

The Empirical Residue.

5./ The suppositions and conceptions resulting from insight commonly are abstract. They abstract from the irrelevant, the insignificant, the negligible, the incidental. They concentrate upon the relevant, the significant, the important, the essential.

But what is the relevant, the significant, the important, the essential? The answer depends immediately upon the insight, or set of insights, grounding the supposing, considering, thinking, defining, formulating. Ultimately, one will say that the answer depends on which insight, or set of insights, is right. But we are not yet ready to tackle ultimate questions. Accordingly, we have to acknowledge, for the present, that the relevant and the irrelevant, the significant and the insignificant, the important and the negligible, the essential and incidental, vary with one's insights. What at one time, one thinks important, later, in the light of fuller insight, one will think unimportant. Inversely, what one used to think insignificant, now one may think significant; and what makes the difference is the advent of further insight.

Still, even for the present, this relative pronouncement is not the whole story. For if we restrict ourselves to the insights possible in mathematics, physics, chemistry, biology, sensitive psychology, and such sciences, then there are elements or components in sensible data and in images that always are regarded as irrelevant, insignificant, negligible, incidental. Such elements or components may be named the empirical residue. They are given as a matter of fact. But they are always disregarded when one concentrates on whatever one happens to think essential.

On four aspects of this empirical residue, something must now be said. They are 1) the individual, 2) the continuum, 3) place and time, and 4) the actual frequency of events.

Individuality

5.1 Individuality pertains to the empirical residue. For whenever we understand anything, we would understand an exactly similar instance in exactly the same fashion. A different understanding would presuppose a difference in the data. It would presuppose the possibility of saying that the previous understanding would do, were it not for this aspect of the object. But, *ex hypothesi*, there is no aspect in which the second object differs from the first, and so there is no possibility of a different understanding. One may learn something new when one turns to the second object; but one automatically learns it about the first object as well.

motor car

Thus, a first motor-car off the assembly line may be understood in terms of certain principles of construction and of operation. A second motor-car, similar in all respects, cannot but be understood in exactly the same fashion.

Nor is the issue changed essentially when one understands instances that are unique. In this case, there is no possibility of apprehending a second object and understanding it in the same manner. But there is the possibility of apprehending the same object a second time: the data in the second apprehension will be similar to those of the first: because the data are similar, the understanding has to be the same. The fact that the similar data are of the same object does not alter the underlying principle that our knowledge is so constituted that similar data have to result in similar insights with the consequence that, what is grasped by insight, is independent of the individuality of the data.

Thus, if the development of all life on this planet were comprehended in a single evolution, there would be no remainder of life on the planet to be understood in either the same or a different fashion. The object would be unique and unparalleled in our experience. None the less, the understanding would consist in grasping principles and laws in the combinations suitable for mastering the enormous ranges of data, while knowledge of the unique instance would consist in observing the data to be understood.

Again, what is grasped by insight, may be named an idea or form emergent in sensible presentations or imaginative representations. But it is one thing to say that grasp of such an idea or form is knowledge of individuality, and quite another to say that within our experience there is found only one instance in which the idea or form can be grasped. If grasp of the idea or form were knowledge of individuality, then the individual would be known by understanding and it would not pertain to the empirical residue. But the mere fact that in some cases there is but a single, observable instance, in which the idea or form can be grasped, provides no evidence for the intrinsic intelligibility of individuality.

In brief, nothing is explained by saying that it is this instance. Inversely, in so far as we grasp explanations, we know not instances but what may or may not be found in individual instances.

Continuity.

5.2 The continuum pertains to the empirical residue. Let us begin with a definition that departs from ordinary mathematical usage to meet our present purpose. A variable, $\underline{x}$, will be said to be continuous in the range, $\underline{a} < \underline{x} < \underline{b}$, if the values of $\underline{x}$ in every part of the range form non-countable infinite sets. Next, a function, $f(x)$, will be said to be continuous in a range if 1) $\underline{x}$ is continuous in the range and 2) for every distinct value of $\underline{x}$ there is a corresponding ~~distinct~~ value of the function. Finally, continuous functions possess a number of distinctive properties; hence, through the verification of these distinctive properties, it may be possible to verify the existence of continuous functions and so conclude to the existence of continua.

Now a continuum, in this defined and verifiable sense (which does not suppose a non-countable infinite set of observations) includes what cannot be counted because it cannot be ordered or systematized. By this inclusion of the non-systematic, a continuum clearly pertains to the empirical residue.

5.3 Place and time pertain to the empirical residue. Space is a continuum of individual positions. Time is a continuum of individual instants. No position is any other. No instant is any other. And of both there are non-countable infinite sets. But the individual and the continuum both pertain to the empirical residue. So also, then, must place and time in their basic aspects.

Hence, when different experimenters, performing the same experiment at different places or times, obtain different results, then no one dreams of explaining the difference in the results by the differences in the place or by the difference in the time. The appeal always is, not to the place, but to something in the place, and not to the time, but to something at the time.

Indeed, if place or time made any difference, then each place and each time would have its own physics, chemistry, and biology. For if place were relevant, the laws in one place could not be the laws in another. If time were was relevant, the laws at one time could not be the laws at another. Further, since places and times are non-countable sets, there would be non-countable sets of different physics, different chemistries, different biologies. Finally, none of the elements of these sets could be ascertained. For one cannot set up a whole physics, or a whole chemistry, or a whole biology, with the observations or experiments made at a point-instant.

However, it is only in their basic aspects that place and time pertain to the empirical residue. A place can be of singular importance, provided that importance rests not on a mere "there" but on a "something there". Such is the importance of the place occupied by a central mass in a gravitational field. Similarly, a time can be of singular importance, provided its importance rests not on a mere "then" but on what happened then. Such is the importance of the initial moment in certain theories in the expanding universe.

5.4 Actual Frequency

Actual frequency pertains to the empirical residue.

The probability of tossing "heads" is 1/2. But in any series of actual tosses, one does not obtain a regular alternation of "heads" and "tails". Between the probability and the actual frequency, there is a divergence. Moreover, this divergence is random. It cannot be reduced to any law or mitigated by any reasonable expectation. It is non-systematic. It is to be known in each case only by actual observation. It too pertains to an empirical residue.

5.5 Significance of the Empirical Residue Let us now recall an initial restriction. The empirical residue was defined as always irrelevant from the viewpoint of insights in mathematics and the natural sciences. Why was this restriction imposed? Quite clearly, because in such a science as the theory of knowledge the notion of the empirical residue attains a systematic significance. For in a study of knowledge one attends systematically, not only to what is concentrated upon in abstraction, but also to what is regularly abstracted from. Theory of knowledge is a higher level science that takes as its materials the whole of the knowledge in other sciences.

Indeed, the theoretical account of the empirical residue is of considerable significance.

It is because insight abstracts from the individual that science is of the universal. It is because science is of the universal, that the observation of similarities is of such great heuristic importance.

It is because insight abstracts from the continuum by proceeding to the limit that the infinitesimal calculus is such a unique and powerful instrument in the construction of theories.

It is because insight abstracts from place and time that principles and laws are independent of place and time and that the expression of principles and laws is invariant with respect to transformations of certain groups of coordinate systems.

It is because insight abstracts from the random divergence of the actual frequency that probability theory has its place among the instruments of scientific knowledge.

Generally, corresponding to each aspect of the empirical residue, there will be a remarkably powerful technique of intelligence in mastering the multiplicity of sensible data. Unfortunately, the discovery of the techniques has to be prior to the determination of the complementary aspect of the empirical residue. For while all aspects of the empirical residue are given on the level of observation, still one can grasp them as pertaining to the empirical residue only by understanding the corresponding techniques.

INSIGHT

CHAPTER II

HEURISTIC STRUCTURES

So far our illustrations of insight have been drawn from the field of mathematics. We have ~~been~~ examined the definition of the circle, the transition from arithmetic to algebra, the distinction between different kinds of infinite sets. It is true that we began from the story of Archimedes' discovery of principles of displacement and specific gravity. But then we were content merely to indicate the more obvious features of insight and made no attempt to analyze the precise nature of the origin and development of scientific knowledge. Such an analysis must now be tackled.

1.1 Similarities of Mathematical and Scientific Insights.

Galileo's determination of the law of falling bodies not only is a model of scientific procedure but also offers the attraction of possessing many notable similarities to the already examined process from the image of the cart-wheel to the definition of the circle.

In the first place, the inquiry was restricted to the immanent intelligibility of a free fall. Just as we ruled out of consideration the purpose of cart-wheels, the materials from which they are made, the wheelwrights that make them, and the tools that wheelwrights use, so also

What is happening? Consider the algebraic procedure that we are generalizing and observe the isomorphism. Where before we said, Let *x* be the required number, now we say, Let "f(x, y, z, t) = 0" be the required function. Where before we noted that, while the minute hand moves over *x* minutes, the hour hand moves over x/12 minutes, now we work out a differential equation that expresses mathematically certain very general features of the data, such as continuity, indestructibility, incompressibility, homogeneity, and so forth. Where before we appealed to the fact that at three o'clock the hour hand had a fifteen minute start on the minute hand, now we turn our attention to boundary conditions that restrict the range of functions satisfying the differential equation.

Restricted Invariance.

2.5 Place and time, no less than individuality and continuity, pertain to the empirical residue. It follows that the function to be determined will hold independently of particular places and times for, as has been seen, particular places and times are, in their basic aspect, continua of individual differences.

Thus, Newton's first law of motion is to the effect that a body continues in its state of rest or of uniform motion as long as no external force intervenes. This law might be regarded as a positive correlation between zero acceleration and zero force. But directly it regards constant velocities and its contention is that such velocities pertain to the empirical residue. If there is an acceleration, mechanical analysis has to assign a corresponding force. If there is no acceleration, then mechanical analysis does not have to bother about assigning any force. Like rest, constant velocity lies outside the range of problems envisaged by Newtonian mechanics. It is a residual feature that needs no positive explanation.

Indeed, there could be no positive explanation of a constant velocity. For it is mere change of place and mere change of time. One can account for change in velocity, and one does so by the law of force. One might account for the conservation of acquired velocity, but that would be, perhaps, a philosophic question rather than a mechanical one. But one cannot assign a positive explanation for every element in change of place for, since places are continuous, since a continuum is a non-countable infinite set of differences, there would be needed a non-countable infinite set of positive explanations for every instance of constant velocity. But a non-countable infinite set of positive explanations is impossible. Therefore, a single explanation has to serve for the whole duration of a constant velocity, and that is provided when one explains the acceleration that terminates in the constant velocity.

However, as is clear from its premise, the point we are making is more general than Newton's first law of motion. The argument rests on the impossibility of a non-countable infinite set of positive explanations. If it may underpin Newtonian mechanics, it may also underpin Maxwell's theory of the electro-magnetic field. Hence, if we may use the technical formulation of the postulate of the Special Theory of Relativity, we may conclude that the mathematical expression of the principles and laws of physics is invariant in form under transformations from one set of coordinate axes to another set moving with a relative constant velocity. (See Lindsay and Margenau, 101 f, 326 ff.).

2.6 *Equivalence.* An even more general heuristic anticipation can be set forth.

The empirical inquirer measures and correlates the results of measurements to reach the functions that relate things directly to one another. There follows a principle of equivalence for all observers.

For, since the function sought relates things directly to one another, the relations of things to observers are omitted. Because the relations of things to observers are omitted, the functions cannot be modified by variations in the relations between the observers and the things. Because there cannot be any such modification, the functions must be the same for all observers.

It is to be noted that the principle of equivalence goes far beyond mere independence of particular places and particular times. Colors as observed vary with the position, velocity, acceleration, of the observer; they vary with the intensity of the light by which he views them; they vary with the condition of his eyes, such as his need of spectacles and his possible color-blindness. But colors as explained by a series of wave-lengths of radiation are necessarily the same for all observers; all conceive them in the same fashion; no one is handicapped by color-blindness.

Now this principle of equivalence represents a property of the direct relations of things to one another. Such a property can be employed as a premise to determine what the relations are. How can such a premise be formulated? A partial formulation is to take the origin and orientation of coordinate axes as representing the observer, and to say that functions, representing principles ~~are~~ or laws, satisfy the principle of equivalence if they remain invariant in form under the group of continuous transformations. For if the observer moves about, he does so in some continuous fashion. But the functions representing laws are independent of any such motion of the observer. And this independence is guaranteed to them by their invariance under continuous transformations.

Such is the postulate of the General Theory of Relativity, which has had some confirmation, and of the Generalized Theory of Gravitation, which as yet has not been put in a form that admits an empirical test.

Certain observations are in order.

First, scalars, vectors, and generally tensors are quantities that may be defined by their transformation properties. Thus, a set of n quantities forms a contravariant vector if they transform according to the same rule as the differentials of the coordinates. A set of n quantities forms a covariant vector if they transform in an opposite manner to the differentials of the coordinates. Contravariant and covariant tensors are sets of n^2 and higher orders of quantities that transform in a more complicated but analogous fashion. Hence, by expressing physical principles and laws in covariant form, automatically there is attained invariance under the group of continuous transformations. [On the tensor calculus, the reader may consult for a brief outline the second chapter of G.C. McVittie's Cosmological Theory, London 1937, Methuen's Monographs on Physical Subjects.]

[ ]
Footnote

Secondly, invariance will be obtained only in so far as there are expressed the relations of things to one another. As soon as equations are made more specific by appealing to observational data of any kind, there is introduced a determination from relations to observers: and then invariance is no longer to be expected. Perhaps this accounts for the fact that in the General Theory of Relativity, the equations remain invariant only as long as the coefficients, g_{ij}, remain in place. See Lindsay and Margenau, p. 368.

g_{ij}

Thirdly, the same consideration seems relevant when one attempts to understand the apparent incompatibility of General Relativity and Quantum Mechanics. As will appear presently, Quantum Mechanics is concerned with observables. It seeks formulations of things in their relations to us while General Relativity rests on the relations of things to one another, and only in its applications turns to relations to us.

Fourthly, the heuristic significance of the principle of equivalence, interpreted as a principle of covariance, is not that it restricts the field of possible laws but rather that it gives a determinate meaning to the empirical investigator's preference for the simplest laws. As A. Einstein has advanced in his autobiography [(Albert

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Einstein, Philosopher-Scientist, ed. P.A. Schilpp, Library of Living Philosophers, 1949 and 1951, New York, Tudor Publishing Company, p. 69.), any law could, perhaps be expressed in covariant form but within the restriction of such a form one can begin by working out the simplest laws and, if they fail, advance to the more complex.

Fifthly, of interest in this connection is Einstein's conviction that data alone are insufficient to guide the constructive efforts of intelligence. There also is needed a formal principle that functions as does the negation of the possibility of a perpetuum mobile in thermodynamics. Such a formal principle Einstein believed he had found in his postulate of invariance, first, in Special Relativity, and then, in General Relativity. (See ibid., pp.53, 57, 69.)

2.7 Summary. Before we turn to the consideration of statistical laws, a summary would seem to be in order.

After noting the similarities between mathematics and empirical insights (1.1) and the differences between them (1.2), we raised the question of the origin and nature of the clues, hints, suggestions that lead up to insight.

As a clue for insight into clues we took the solution of a simple algebraic problem (2.1) and proceeded to generalize.

What is to be known, when the insight occurs, is anticipated by the mere fact of inquiry and is named the "nature of ....", the "such as to...", the "sort of thing that. . . ".

But similars are similarly understood. Hence, the "nature of . . .", may be specified by means of a classification based on sensible similarity; and when insight occurs, this preliminary classification will yield place to a systematic account that speaks of things, not in terms of their relations to our senses, but in terms of their relations to one another. Thus, the "nature of..." is replaced by the more precise anticipation of an unspecified correlation to be specified, of an indeterminate function to be determined (2.3).

Now functions can be determined, not only by the empirical process of reaching formulae that all known measurements satisfy, but also by appealing to quite general considerations and arguing from them to differential equations which restrict the group of possibly relevant functions. Quite obviously, both procedures can be combined and commonly are combined to obtain a scissors-like action that approaches a solution both from above and from below (2.4).

Further, when differences form a non-countable infinite set, as is the case with place and time, there cannot be a distinct explanation for each element of difference. Hence constant velocity has to be regarded as residual and, in fact, it is so regarded in Newton's first law of motion. More generally, the mathematical expression of principles and laws has to be invariant under transformations between inertial systems in accordance with the postulate of Special Relativity (2.5).

Indeed, inasmuch as principles and laws express the relations of things to one another and omit all reference to the relations of things to observers, it follows that the mathematical expression of principles and laws must be invariant in symbolic form under continuous transformations (2.6).

Finally, one may add that these considerations supply only an abstract scheme. In concrete inquiry they are employed not singly but together. As a science develops, all that already is known serves to render more determinate and precise the general heuristic anticipations that have been outlined.

Statistical Heuristic Structures

3.0 The fact of inquiry is an anticipation of something to be known by understanding. Hitherto, only one type of such anticipation has been considered, namely, the anticipation of a correlation, a function, a law, a system. The investigator measures, plots his results upon a graph, and expects to find a smooth curve or formula that will be satisfied, not only by the measurements he has made, but also by all the relevant measurements that he or anyone else ever will make.

Now it is well to encourage investigators in that expectation, to tell them that, if they do not discover any law, then, perhaps, they are measuring the wrong things, that they are not excluding some extraneous influence, that if only they are dogged enough, some day someone will discover the relevant correlation, function, law.

Still, encouragement must not be carried to the point of deception. As we have seen, there is an empirical residue, and the insight relevant to it consists in grasping, not the system to which it conforms, but its ultimately non-systematic character. Hence, with respect to an aggregate of data or measurements, the anticipation implicit in the fact of inquiry is not a single assertion but rather a disjunction. The anticipation is, not that there must be some correlation to be grasped, but that either there is such a correlation or else there is not. The positive member of the disjunction has been considered in the foregoing account of anticipations of the systematic, and now we must endeavor to clarify the meaning of anticipations of the non-systematic.

3.1 The Non-Systematic.

To reach a classical correlation, function, rule, law, theory, system, there is needed an initial insight into some particular case. By that insight one may master an indefinite multitude of exactly similar cases. Still such universality is not enough The significance of the initial insight is that it can lead to further insights that master ever more dissimilar particular cases until eventually one reaches a general case and brings under one's control a definable range of particular cases. So Galileo's understanding of the free fall regarded, not bodies of some determinate size, shape, and weight falling at some fixed inclination from the vertical, but bodies of any size, any~~x~~ shape, any weight, falling at any inclination from the vertical.

Now a heuristic anticipation of the non-systematic implies, not a denial of the possibility of concrete insight into particular cases, but a denial of the possibility of the abstract generalization that subsumes a range of particular cases under a general case. In other words, the non-systematic is not to be identified with the non-intelligible. While the non-systematic excludes the generality of classical correlations, functions, rules, laws, theories, systems, it need not exclude the intelligibility to be reached by inspection and insight into particular cases.

For example, in a particular case, dice may be cast from a determinate receptacle in a determinate manner upon a determinate surface; sufficient information on the case could be attained with the help of a slow-motion film; insight could analyze the total movement into a sequence of mechanically homogeneous stages; each stage could be subsumed separately under known laws of motion, gravity, air resistance, impact, friction, and elasticity; and the total movement would be no more than the sequence of the stages. Still, dice can be cast from any sort of receptacle, in any manner whatever, upon any type of regular or irregular, fixed or moving surface. There would be no point in attempting to repeat the above laborious procedure for the infinity of particular cases; and if casting dice is a non-systematic, there exists no general case of the classical type to provide an alternative to a pointless repetition of merely particular investigations.

3.2 Actual Frequency.

Where classical generality fails, statistical generality may be sought.

Let us say, then, that there an exists an ~~actua~~ actual frequency if, from some determinate antecedent, O, there always follows one and only one of the alternatives, P, Q, R,.... For in any n occurrences of the antecedent, O, the alternative, P, will occur on a determinable p occasions, Q on q occasions, R on r occasions, etc. Accordingly, the actual frequency of P in times, say r, etc. Hence, the actual frequency of P in a given n occurrences of O will be p/n, the actual frequency of Q will be q/n, the actual frequency of R will be r/n, etc., so that necessarily

$$n = p + q + r + \ldots\ldots$$

Finally, these actual frequencies will be non-systematic if it is not possible to define an Op, Oq, Or, P', Q', R', such that P' always follows Op, Q' always follows Oq, R' always follows Or, etc., so that the indeterminateness of the alternatives is eliminated.

It is to be noted that when a set of alternative consequents has been defined, then it is possible by combinations to construct further sets of alternatives. Thus, one can consider the actual frequency of the combination "either P or Q", or of the combination "P on a first occasion and Q on the second occasion", etc., etc.

One may add at once that the actual frequency of a number of alternatives taken together is the sum of their actual frequencies taken separately. Thus, the actual frequency of "either P or Q" will necessarily be (p + q)/n. Similarly, the actual frequency of the total set of alternatives will necessarily be n/n or unity.

A Generic Notion of Probability.

3.3 Let us now generically define a probability as the proper fraction from which actual frequency does not diverge systematically.

The definition posits an ideal proper fraction, which it names a probability. It admits that this ideal proper fraction will not be coincident with actual frequencies. It denies that the divergence between the ideal and the actual will be systematic.

Suppose, for instance, that the probability of casting a "six" with a single die is 1/6. Then, on the first six throws, a "six" may occur twice, on a second, once, on a third, not at all, etc... The actual frequency hops about in random fashion while the probability always remains the same 1/6. There is then a divergence between the actual and the ideal. But this divergence is non-systematic so that the difference between the actual and the ideal cannot be reduced to any rule or law.

Certain clarifications are in order.

First, the reason for the definition is, perhaps, obvious enough. Actual frequencies are non-systematic; they vary from case to case; and their variation is not subject to any rule or law. But a probability is an ideal fraction; it is the same for every case of a given kind; it is the representative of the universal, abstract, necessitating, systematizing tendencies of understanding. Hence, if probability and actual frequency coincided, then either both would be systematic or both would be non-systematic. If they diverged and the divergence were systematic, then the actual frequency would have to be the systematic resultant of the systematic probability and the systematic divergence from probability. One meets the requirements of the problem only if 1) the actual frequency is non-systematic, 2) the probability is somehow systematic, 3) the actual frequency may diverge non-systematically from the probability, and 4) the actual frequency cannot diverge systematically from the probability.

Secondly, it follows that the probability of a set of alternatives is the sum of the probabilities of the alternatives taken singly. For, as we have seen, the actual frequency of such a set is the sum of the actual frequencies of the members of the set (3.2) and, moreover, there cannot be a systematic divergence between actual frequency and probability. But there would be such a systematic divergence if the probability of the set were not the sum of the probabilities of the members of the set. Accordingly, one must deny the consequent and its antecedent to affirm that the probability of a set of alternatives is the sum of the probabilities of the alternatives taken singly.

Thirdly, a probability is not the mathematical limit of a series of actual frequencies. For a series of terms tends to a mathematical limit inasmuch as divergence from that limit can be made as small as one pleases. But actual frequencies do not converge upon probability. They hop about at random. They approach the probability only to recede. Instead of converging, they diverge. But they cannot make their divergence effective, for they cannot get any system into it.

Fourthly, though a probability is not a mathematical limit, there are unobjectionable assumptions that may be introduced so that the non-systematic divergence of probability becomes virtually equivalent to the convergence characteristic of the mathematical limit.
(See Lindsay and Margenau, pp. 165 ff.)

Fifthly, our procedure will be to distinguish two radically different meanings of the term, probability. As defined, probability is an ideal proper fraction from which actual frequencies can diverge but not systematically. However, one also speaks of the probability of opinions and then one does not mean that there is some fraction relevant to the opinion. What is probability in this second sense and what is its relation to probability in the first sense, are questions that must for the moment be postponed.

3.4 Specific Differences. It is one thing to calculate the probability of throwing a "four" with a single, unbiassed die, another to make the same calculation when a pair of dice are used, and a third, to do so when the dice are "loaded". In all three cases there is the same generic element: actual frequency diverges non-systematically from the proper fraction named probability. But this genus divides into three distinct species, and the basis of the division resides in the manner in which probability is determined.

The first species is equiprobability. Its conditions are that 1) when an antecedent, O, occurs, then there occurs one and only one of a set of n alternatives, and 2) there is no systematic favoring of any of the n alternatives. From the conditions it follows that the probability of the occurrence of any given alternative will be 1/n. For were the probability some other fraction, say, a/n, where a is less or greater than unity, then that alternative could not diverge systematically from a/n and so must suffer systematic discrimination, if a is less than unity, or receive systematic favoring, if **a** is greater than unity.

The second species is a derivative of the first. Its conditions are that 1) when an antecedent, O, occurs, then there occurs one and only one of a set of **n** alternatives, 2) there is a systematic favoring of some alternatives, but 3) this systematic favoring can be reduced to a case in which there is no systematic favoring.

Thus, when a pair of dice are cast, there are eleven possible results, of which some regularly occur more frequently than others. However, this favoring can be eliminated by considering the thirty-six alternatives constituted by combining each of the six faces of one die with each of the six of the other. No one of thirty-six alternatives is favored in any systematic manner, and so the second species is reduced to the first.

The second species of probability is investigated at length by applying the mathematical theory of combinations. The basic formula assigns the probability, P, of **r** successes in **n** tries, when **p** is the probability of one success in one try. This formula is worked out in any suitable text and along with it the reader will find the approximations developed by Laplace, Poisson, and Gauss.

The third species does not admit reduction to the first or to the second. There is an antecedent followed by one and only one of a non-systematic set of alternatives. But one cannot settle by inspection what the alternatives are; and their respective probabilities neither are equal nor are reducible to the case of equiprobability.

Thus, when dice are loaded, some combinations might never occur; moreover, the occurrence of any given face of a loaded die is not equal in probability with the occurrence of any other face, for there is some systematic favoring.

The third species may be described as involving a systematic element which, however, does not succeed in completely dominating the results. There is a systematic element, otherwise the alternatives would be equiprobable. But the systematic element does not succeed in dominating the results, for they are found to be non-systematic.

To meet the problem set by the third species, the relevant technique would seem to be 1) to loosen the heuristic anticipations for dealing with data that can be reduced to system and 2) to compensate for this loosening by introducing probabilities in place of precise predictions.

What would such loosening be? First, anticipations of the systematic are 1) that the data will satisfy some one law or function, 2) that this function will be a solution of the differential equations that represent general features of the problem. Secondly, these anticipations can be loosened. Instead of expecting one function to cover all the data, one may expect a series of eigenfunctions, say ψ_λ, and a corresponding series of eigenvalues, say p_λ. Again, instead of expecting the single function to be a solution of a differential equation, one may expect the eigenfunctions and eigenvalues to be the solutions of an operator equation, say,

$$P\psi_\lambda = p_\lambda \psi_\lambda$$

where P is the operator, that is, a mathematical entity that changes one function into another.

What is the compensating? The foregoing yields a set of observables, the eigenvalues, p_λ. Those that occur will possess some probability, else they would not occur; and they will not possess more than probability, else a systematic solution would work. There exists then some state function from which the probabilities can be calculated; and one may expect the eigenfunctions to lead to the determination of the state function, for if they succeed in selecting the observables with some probability, they should be able to contribute to the determination of the respective probabilities.

Is this guess-work? Certainly, it is not a rigid deduction. On the other hand, it is not purely arbitrary. It is the fruit of an insight based upon clues where, as is always the case, the insight takes one beyond the clues. There must be some loosening of systematic anticipations, for the data dealt with are only partially under the influence of what one might name a systematic component. There must be some compensation for this loosening, else there would be no conclusions at all. But the exact course of the loosening and the compensating is guided by insights into mathematical possibilities and, however strangely, the resulting postulates of Quantum Mechanics have proved highly successful.

For an insight into the Operators P. see G. Temple, The General Principles of Quantum Theory, Methuen's Monographs on Physical Subjects, London 1951.

3.5 Summary.

Classical method is not content with mastery of particular cases but goes beyond them to the abstract generality expressed in correlations, functions, laws, theories, systems. However, there is an empirical residue; particular cases can consist in coincidental manifolds of distinct instances of general cases; and corresponding to such coincidental manifolds there is no general case of the classical type. Still this negation of systematic generality is not the negation of all generality. For if one supposes data to be involved in the non-systematic, one cannot suppose that they diverge systematically from ideal norms.

Among such ideal norms the most familiar is the probability of the occurrence of one of alternative possibilities; and the mode of its determination also supplies its subdivision. If there is no systematic favoring of any of the alternatives, there is equiprobability. If there is systematic favoring that can be reduced to equiprobability, Newton's formula becomes the relevant anticipation. Finally, when there is systematic favoring that cannot be reduced to equiprobability, than some special axiomatic structure has to be involved.

There is, then, a statistical heuristic structure and it complements classical structure. In any selected field of inquiry, experiments are performed, measurements are made, and the results are tabulated. In so far as the general intelligibility of the measurements is systematic, classical procedure is relevant. In so far as the general intelligibility of the measurements is not systematic, a probability function is to be sought. Finally, since antecedently the general intelligibility of measurements may be either systematic or non-systematic, a general theory of measurements must envisage both alternatives. May I ask whether this requirement, rather than particular hypotheses on the accuracy or the distorting effect of measuring, can be regarded as the ultimate basis of the insight into operators that is offered by G. Temple in The General Principles of Quantum Theory [Methuen's Monographs on Physical Subjects, London 1951]?

May further suggestions be made? As long as physicists were engaged in introducing ever more complex modifications of Bohr's image of the atom, they were endeavoring to mount through particular cases to the general case. When they decided to limit their equations to observables (i.e., variables admitting experimental control), they surrendered not generality but systematic generality. Again, in so far as Quantum Theory may be said not to offer insight into particular cases, it suffers on that lower level a perhaps irremediable incompleteness; on the other hand, interpreted as a statistical theory, it possesses fully the completeness of the non-systematic general case.

If such suggestions are to be tried out, it is not to be forgotten that our account of probability supposes an explicit advertence to insight, that other accounts do not, and that the other accounts not only possess the field but also penetrate the interpretation of scientific results. Only a critical and creative effort, meticulously separating methodological assumptions from scientific hypotheses, can determine adequately the relevance of the present analysis to the problems in which scientists are involved; or in the simpler words of Einstein's rather celebrated remark, the cognitional theorist has to attend, not to what scientists say, but to what they do.

Appendix to Chapter II.

ON THE USE OF THE TERMS "CLASSICAL" AND "STATISTICAL"

"mechanical"

may

In ordinary usage, "classical" and "statistical" are not opposed. The opposite to "classical" is "quantum", and the opposite to "statistical" is "mechanical". This usage may be illustrated by the fourfold classification of 1) classical mechanics (Newton), 2) classical statistics (Boltzmann), 3) quantum mechanics (Schrödinger, Heisenberg), and 4) quantum statistics (Bose-Einstein, Fermi-Dirac).

The trouble is that this fourfold classification seems incomplete. For relativity mechanics is opposed to classical mechanics and, while special relativity enters into combination with quantum mechanics (Dirac), general relativity seems as opposed to it as Einstein himself. Further, if these complications are not to be neglected, it is necessary to go behind the terminology to a systematic conception of the conceptions entertained by interpreters of physical theory. As is obvious, however, the purpose of this appendix is not to expound and to justify a systematic view but simply to clarify the linguistic usage that we have found convenient by contrasting its assumptions with the assumptions that seem to underlie more common modes of speech.

From our viewpoint, then, the fundamental disjunction regards the interpretation of laws of the Newtonian and Einsteinian type. Such laws will be said to be interpreted concretely if they are taken to relate imaginable terms. The same laws will be said to be interpreted abstractly if they are taken to relate terms that are defined implicitly by the laws themselves.

On the first alternative of concrete interpretation, the law is completely determinate in principle. It is true enough that the law is expressed by a mathematical formula of wide generality and that further determinations will have to be added before any application to concrete instances can occur. It also is true that the further determinations cannot be deduced from the law as a mathematical or as a physical formula. But on concrete interpretation the law is not simply a physical formula: it relates imaginable terms; and because terms are imaginable inasmuch as their various dimensions are assignable, it follows that for concrete interpretation the law is fully determinate in principle.

However, those that accept the first alternative split into two groups. The first group not only affirms concrete interpretation but also affirms that concretely interpreted laws of the Newtonian type exist. The second group agrees with the first in admitting concrete interpretation but differs from it by affirming that, if any such laws seem to be verified, the verification is mere macroscopic appearance. The agreement and difference of this first and this second group seem to me to correspond to the agreement that unites and the difference that separates ordinary conceptions of classical statistics and quantum mechanics.

On the second alternative of abstract interpretation, the foregoing debate is replaced by a distinction. Concretely interpreted laws of the Newtonian and Einsteinian type are resolved into abstract and concrete components. The abstract component is the verified correlation of implicitly defined correlatives. The concrete component is the schematic or non-schematic situation.

The abstract component is determinate but not fully determinate. It is determinate in its own abstract order as an element in abstract system. But it becomes fully determinate only when it is applied successfully to concrete situations. Such application calls for two further types of information: first, one must know which laws in what combination are relevant to the given situation; secondly, one must know what numerical values are to be substituted for the variables and general constants of the abstract laws.

Now while there are well-known difficulties in obtaining accurate numerical values by measurement, a far more radical difficulty arises when one does not know exactly which combination of laws is relevant to a given situation, for then one is unable to go about the task of measuring in any orderly and economical manner. Fortunately, however, there do exist schematic situations in which a happy constellation of circumstances and an appropriate combination of laws have the encouraging implication that the same laws will be applied over and over again in an indefinite sequence. Such, for example, is our planetary system, which has provided the most striking instances of accurate deduction and long-term prediction.

Unfortunately, there also are non-schematic situations. Then the task of applying abstract laws to concrete situations is at the mercy of circumstance, and the relevant circumstances form a diverging and scattering series of ever more numerous and more remote conditions. For example, a planetary system has a beginning and may come to an end; either event can occur only once for any given system; and then it can occur in any of a notable range of different manners.

Still the existence of non-schematic situations, so far from blocking human intelligence, gives it a new impetus. Statistical investigation becomes the key to an account of the emergence and survival, the numbers and distribution, the differentiation and development of schematic situations. Classical anticipation of the systematic and statistical anticipation of the non-systematic cease to be disparate alternatives. They become complementary techniques in gaining insight into a universe in which the thrust of probability generates from the non-schematic ever more numerous and more developed instances of the schematic.

Accordingly, our contrast between classical and statistical rests not on current issues but on their transposition. On the basis of cognitional analysis the opposition between determinism and indeterminism is sublated in favor of a more comprehensive structure. Classical laws are reinterpreted so that Einstein's differential equations are regarded, not as statements about events at point-instants, but as mathematical expressions of the abstractness of classical laws. Statistical laws are reinterpreted so that indeterminacy has its root in the abstractness of classical laws, its factual ground in the existence of non-schematic situations, and its significance in the type of explanation associated not with the name of Laplace but with the name of Darwin.

observations to assemble into the conditions at some nth remove for some specific event.

The Non-Systematic Aggregate of Diverging Series.

6.53 Unfortunately, there is no system to the aggregate of concrete patterns of diverging series of conditions for all kinds of events. Full and exact knowledge of all classical laws assures only a systematic unification of the laws. Such a systematic unification is not an imaginative synthesis. On the other hand, each of the concrete patterns of diverging series is an imaginative synthesis. It follows that singly and together these concrete patterns are non-systematic, for the totality of systematic relations is included in the totality of abstract laws.

Now this general argument can be set forth in more concrete fashion inasmuch as the reader can be offered the materials for two insights. The first insight will be a grasp of the non-systematic in a familiar case. The second will be a grasp of the same lack of system in the aggregate of concrete patterns of diverging series of conditions.

The familiar case may be defined by the question, How many ways are there to cast a "five" with a single die? One might attempt to answer this question empirically. One would get a high-speed camera, suitable lighting, a transparent box, and proceed to take pictures. Next, one would study the pictures of all cases in which a "five" was thrown and calculate the linear and angular momenta in each movement of the die. The more diligent one was, the greater would be the number of known distinct manners in which a "five" can be thrown. But no matter how great one's industry, one could hardly arrive at the point where one could say one knew all of the ways in which a "five" could be thrown with this die from this box on this surface. Accordingly, one would shift to an *a priori* method. One would work out a formula that gave the maximum and minimum initial momenta for the last stage of a throw, and the formula would contain constants that received different numerical values for different surfaces and different dice. From the formula one could list all the possible combinations of specifications for the last stage of throwing a "five". By introducing a convenient supposition to prevent the list from containing a non-countable infinite multitude of cases, one could proceed to the second last stage of the process: it would end in any of the manners in which the last could begin; and a further formula would enable one to assign a multitude of ways in which the second last could begin for each way in which the last could begin. With this multitude of multitudes on one's hands, one could turn to the third last stage, and so forth.

Now we happen to know that throwing a "five" is a non-systematic process. While each movement in the process is determinate, while the relations between successive movements are determinate, still these relations depend upon concrete circumstances no less than abstract law. ~~cannot be subsumed under any rule or law~~. The purpose of the preceding paragraph was, not to show that throwing a "five" is non-systematic, but to grasp in that instance of the non-systematic some of its distinctive characters or symptoms. Our first discovery, then, was that an empirical method of observation and analysis could reveal a great number of ways in which the result might occur, but it offered no promise of providing a complete list of all the ways. Our second discovery was that an *a priori* method yielded an unmanageable variety of different combinations of distinct alternatives. Even though distinct stages of the process were summed up in formulae, still every possible combination of numerical values satisfying the formulae offered a different alternative, and combinations of these alternatives defined the different ways.

Let us now turn to the second insight. Consider any event, X, and let it be defined as a determinate numerical value of some variable in some classical law.

Next, consider all the laws in which this variable occurs, and list all the alternative combinations of numerical values for the other variables in these laws when the event, X, is occurring.

Thirdly, consider the different manners in which each of the alternative combinations may be approached. Thus, if there are *n* variables involved and they may have the numerical values, *a*, *b*, *c*,.... when the event, X, is occurring, then the *a*, *b*, *c*, ... specify one of the alternative combinations. Now there are different combinations of rates of change in these variables, such that the rates of change are compatible and, as well, they bring the variables to the values, a, b, c, ... A complete list of such combinations of rates of change, first, when the rates are regular, and secondly when they are not, would serve to define the different approaches to one of the alternative combinations.

Fourthly, repeat the foregoing performance for all kinds of events. Then, one will have worked out all the manners in which one may approach at all possible combinations of rates of change all the alternative combinations of numerical values for the other relevant variables when each variable in each law assumes every possible numerical value.

Fifthly, by comparing different processes, one can draw up a list of incompatible events.

Sixthly, by combining compatible processes in all possible manners, one can Cast ~~abstract~~ diverging series of positive conditions for all kinds of events to as many removes as one pleases.

Perhaps this is enough. One is working out a plan of setting up an unmanageable variety of different combinations of distinct alternatives. The intelligent procedure in dealing with such combinations of alternatives is to acknowledge their non-systematic character and turn to the calculation of probabilities. For an a priori method of working out diverging series of conditions yields the concrete patterns that occur, not only in this visible universe, but also in every possible universe subject to the same laws. On the other hand, an a posteriori method would be both impracticable and inconclusive.

Summary

6.54 Such is the argument in the general case.

Classical laws hold in the concrete only if conditions are fulfilled. To invoke the same or different laws to show that conditions will be fulfilled, merely sets up a diverging series of conditions. The further one goes back along the series, the more numerous become the conditions and the more they are dispersed not only in space but also in time. Even if one knew the patterns of the diverging series, and the fulfilment of all conditions at some nth remove, the only possible deduction would be in virtue of the inverse, converging series. Finally, such patterns form a non-systematic aggregate: they are an enormous series of different combinations of distinct alternatives: their intelligibility is reached, not by working them out in detail, but by acknowledging their non-systematic character and turning to probabilities.

The Possibility of Accurate Prediction.

6.55 However, besides the foregoing general case, there is also a ~~number of~~ particular case~~s~~. In the last analysis, ~~they~~ it reduces to the general case. But the last analysis is not reached at once and, in the meantime, there is the possibility of the accurate deduction and prediction of fully determinate events. Accordingly, we have to define the particular case, show how it escapes the logic of the diverging series of conditions, and finally argue that this escape is never complete.

The particular case will be named a scheme. Its abstract or theoretical component is some classical law or combination of laws, such that there arises a mutual fulfilment of conditions. Its concrete or factual component is such a conjunction of things or events that, in virtue of the law or laws, the conjunction leads to another, the other leads to a third, the third leads to a fourth, until eventually the initial conjunction recurs. Such schemes may be extremely simple or extremely complex. They may involve any number of intermediaries or, in the case of the straight-forward continuity, none at all. Moreover, schemes may be combined, so that all will function if any one or two or n function. Finally, schemes may emerge in a conditioned series, such that the later become possible when the earlier are functioning.

The conspicuous example of the schemes of recurrence is of course the planetary system. But the whole of nature seems full of oscillations, rhythms, alternations, recurrences, from the elementary processes of physics to the technological, economic, and political inventions and routines of man. Finally, when such patterns of recurrent activity are submitted to analysis, they are found to involve the two elements of a scheme, the theoretical component of interrelated laws and the factual component of a conjunction that through the laws brings forth its own recurrence.

ponent

Clearly, such schemes do not suppress the principle that no event is unconditioned. Nor do they prevent each event from having many conditions. None the less, though the diverging series of conditions remains, it has been brought to heel. For the scheme itself takes care of its positive conditions, all of which are included within classes of events, and every event within the classes keeps recurring because the others do in a perpetual vicious circle.

There is, then, an escape from the unpredictability implicit in the diverging series of conditions. Were astronomers merely in possession of full and exact knowledge of all natural laws, they still would be stuck with their 3-body problem, that is, with the task of finding a general solution to the problem of determining the trajectories of three bodies when their initial positions and momenta were given. In fact, astronomers operate in the light of an imaginative synthesis; Ptolemy's mistaken imaginative synthesis yielded fair results; Copernicus' simpler imaginative synthesis combined with a more accurate knowledge of laws enables men to predict with remarkable accuracy the movements, not merely of three bodies, but of the sun, the planets, their satellites, the comets, and even asteroids.

Still, this escape is not complete. The periodicity of our planetary system offers no guarantee against internal disruption of its members or against the intrusion of some external body like a bull into our china shop. The planetary system secures its own perpetuity only if certain negative conditions are fulfilled, and over those negative conditions it exercises no control. Moreover, just as the planetary system is not a proof of its own survival, so it is not the ground of its own emergence. A scheme is a matter, not merely of a combination of laws, but also of a happy conjunction of things or events. That conjunction has to take place before the scheme can begin to function, and so the scheme has its origin in a combination which it did not generate.

Now one might like to suppose that, just as there are schemes, so too there is an over-all scheme, an ultimate imaginative synthesis, on which there could be based accurate predictions of the emergence and survival of lesser schemes. Such would be the affirmation of mechanist determinism. But, as we have seen, complete and exact knowledge of all laws would include a systematic unification of laws without involving an imaginative synthesis either of the concrete unfolding of this universe or of any other subject to the same laws. Moreover, an over-all scheme would have not only a theoretical component, constituted by laws in combination, but also a factual component, constituted by an initial conjunction that the over-all scheme itself could not bring about. Finally, the issue before us is to be settled, not by what one might like to think, but by the evidence; and the evidence is that the concrete, historical unfolding of this world process involves a conspicuous use of the statistical techniques of large numbers and long intervals of time. It seems to follow that the over-all intelligibility of our world process is, not in accord with the assumptions of mechanist determinism, but in accord with some different view that assigns a due place to statistical laws. After all, machines are constructed and function within political, economic and technological schemes, and ~~rich~~ such schemes emerge, survive, and are superseded without systematic divergence from the probabilities.

The Indeterminacy of the Abstract.

6.56 We have been endeavoring to indicate in precise terms both the indeterminacy of the abstract and the consequent statistical residues.

In brief, the indeterminacy of the abstract is the indeterminacy of the blanket proviso, "other things being equal". Classical laws are said to hold in the concrete, provided other things are equal, but no one specifies what the other things are or in what their equality consists.

There is good reason for this omission. For a fully determinate event in the general case depends upon the fulfilment of a diverging series of positive and negative conditions. The conditions at each remove in the series not only become more numerous but also scatter in space and time. Finally, the patterns of such diverging series form an enormous, non-systematic aggregate.

It is true that there are schemes of recurrence. Granted any of a long series of suitable initial conjunctions, the operation of classical laws will tend to repeat the initial conjunction indefinitely. Still, there is only a tendency and not an absolute necessity, for here too there rules the blanket proviso, other things being equal or unequal but negligibly so. Nor is there any evidence to support the affirmation of some over-all scheme to regularize the emergence and the survival of lesser schemes.

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The general case, then, is the universal case. In the last analysis, events depend upon a non-systematic aggregate of patterns of diverging series of conditions. Because that aggregate is non-systematic, it is a residue abstracted from by the totality of classical laws. Because the non-systematic is the premise of statistical inquiry, this residue may be named statistical. Hence, the canon of statistical residues may be said to affirm the non-systematic character of the aggregate of patterns of diverging series of conditions that govern concrete events.

A Mathematical Analogy

6.57 A mathematical analogy may exist. For combinations of differential equations are likely to be soluble only through the introduction of special suppositions and, even then, only by a method of approximations. Hence, if one said that classical laws corresponded to differential equations, that concrete problems demanded combinations of such equations, and that the totality of certain of special suppositions and approximate solutions was non-systematic, one would have in the field of mathematics an analogy to the canon of statistical residues.

The General Character of Statistical Theories.

6.6 Finally, the canon of statistical residues, in conjunction with the other canons of empirical method, makes it possible to complement our account of the notion of probability (Chapter II, #3) with a derivation of the general characteristics of statistical theories.

6.61 Events. First, statistical theories will deal

4.1 The Elementary Paradox

Let (x_1, t_1) and (x_2, t_2) be the coordinates of a pair of point-instants, P and Q, in a reference frame, K.

Let (x'_1, t'_1) and (x'_2, t'_2) be the coordinates of the same pair of point-instants in a relatively moving frame, K', and let them from this view-point be named, P' and Q'.

On the Lorentz-Einstein transformation, writing

$$H = 1/(1 - u^2/c^2)^{\frac{1}{2}}$$

one relates the coordinates by the equations

$$x'_1 = H(x_1 - ut_1) \qquad (1)$$

$$x'_2 = H(x_2 - ut_2) \qquad (2)$$

$$t'_1 = H(t_1 - ux_1/c^2) \qquad (3)$$

$$t'_2 = H(t_2 - ux_2/c^2) \qquad (4)$$

Now consider two particular cases. So far, P and Q are any point-instants whatever; but in our first particular case we suppose that P and Q are the simultaneous positions of the ends of a standard measuring rod in the frame, K. Since the length of the rod is unity, and since the positions are simultaneous, we have

$$x_1 - x_2 = 1 \qquad (5)$$

$$t_1 - t_2 = 0 \qquad (6)$$

By subtracting equation (2) from (1) and equation (4) from (3) and substituting the values from equations (5) and (6), we have

$$x'_1 - x'_2 = H \qquad (7)$$

$$t'_1 - t'_2 = -Hu/c^2 \qquad (8)$$

so that, clearly, a unit length between simultaneous positions becomes on transformation a length that is not unity between positions that are not simultaneous.

In our second particular case, we suppose that P and Q are the point-instants of successive seconds in a standard clock stationary relative to the frame K. Clearly,

$$\underline{x}_1 - \underline{x}_2 = 0 \qquad (9)$$

$$\underline{t}_1 - \underline{t}_2 = 1 \qquad (10)$$

whence, as before, by appealing to equations (1) to (4) and by substituting from (9) and (10), one obtains,

$$\underline{x}'_1 - \underline{x}'_2 = - \underline{Hu} \qquad (11)$$

$$\underline{t}'_1 - \underline{t}'_2 = \underline{H} \qquad (12)$$

so that a distance that is zero has been transformed into a distance that is not zero, and a time that is unity has been transformed into a time that is not unity.

Still, though distances and times are relative to reference frames, the four-dimensional interval is invariant. Let us name the interval, $\underline{s}$, where

$$\underline{ds}^2 = \underline{dx}^2 - \underline{c}^2\underline{dt}^2 \qquad (13)$$

and in the present cases

$$\underline{s}^2 = (\underline{x}_1 - \underline{x}_2)^2 - \underline{c}^2(\underline{t}_1 - \underline{t}_2)^2 \qquad (14)$$

On substituting from equations (5) and (6), one finds that the interval of the rod in K according to the account in K is unity. Likewise, on substituting from equations (7) and (8), one finds that the interval of the rod in K according to the account in K' is unity. Again, on substituting from equations (9) and (10), one finds that the interval of the clock in K

according to the account in K is ic $\left[i = \sqrt{-1}\right]$.
Likewise on substituting from equations (11) and (12), one finds that the interval of the clock in K according to the account in K' is also ic.

Thus we have arrived both at the elementary paradox and at its solution. The elementary paradox arises from the contrast of equations (5) and (7) and again from the contrast of equations (10) and (12). The first contrast shows that the length of a rod in K is unity on the account in K but on the account in K' is greater than unity; and if K' finds a unit rod greater than unity, it seems to follow that his own rod is shorter. The second contrast shows that the length of a standard duration in K is unity in the account in K but is greater than unity in the account in K'; and if a unit of duration in K is found to be greater than unity in K', it seems to follow that the unit in K' must be shorter.

However, if we began from rods and clocks in the system, K', we could establish the opposite conclusions with equal validity; for then it would seem to follow that the shorter units were in the system, K. Such is the elementary paradox.

What the paradox over-looks is the fact that, in the context of Special Relativity, one is not dealing with rods that are merely spatial or with clocks that are merely temporal. For, as has been seen, a standard rod determines an invariant four-dimensional interval of magnitude, unity; and a standard clock determines an invariant four-dimensional interval of magnitude, ic. Rods that determine an invariant four-dimensional interval must have a temporal component, and clocks that determine an invariant four-dimensional interval must have a spatial component.

Indeed, as appears from equations (5) and (6), in the reference frame, in which a rod lies between simultaneous point-instants, the invariant interval has a spatial component of magnitude, unity, and a temporal component of magnitude, zero. As appears from equations (7) and (8), in other relatively moving reference frames, the same rod determines the same four-dimensional interval, which, however, now has a spatial component of magnitude, $\underline{H}$, and a temporal component of magnitude, $-\underline{H}\underline{u}/\underline{c}^2$. Concomitant with the variation of the spatial components, there is a variation of the temporal components. The rod in K by the account in K lies between simultaneous point-instants. The same rod in K by the account in K' lies between non-simultaneous point-instants. The spatial and temporal components, say $[1, 0]$, transform to spatial and temporal components, $[\underline{H}, -\underline{H}\underline{u}/\underline{c}^2]$. Inversely, the rod in K' by the account in K' will lie between simultaneous point-instants. But the same rod in K' by the account in K will lie between non-simultaneous point-instants. In this case, spatial and temporal components, $[1, 0]$, transform to spatial and temporal components, $[\underline{H}, \underline{H}\underline{u}/\underline{c}^2]$, for the sign of the relative velocity, $\underline{u}$, changes.

Again, as appears from equations (9) and (10), in the reference frame, in which the beginning and the end of a standard duration occur in relatively the same position, the invariant interval of magnitude, $\underline{ic}$, has a spatial component of magnitude, zero, and a temporal component of magnitude, unity. As appears from equations (11) and (12), in other relatively moving frames of reference, the same duration determines the same invariant interval, which, however, now has a spatial component of magnitude, $-Hu$, and a temporal component of magnitude, H. Again, there is concomitant variation of spatial and temporal components. A standard duration in K by the account in K has components [0, 1]; the same duration in K by the account in K' has components [$-Hu$, H]. Inversely, a standard duration in K' by the account in K' will have components [0, 1]: but this duration in K' by the account in K will have components [Hu, H].

The elementary paradox results from a cumulation of oversights. It disregards the invariant interval fixed by any rod for all reference frames and the invariant interval fixed by any clock for all reference frames. It disregards four accounts of two rods to consider only two rods, and it disregards four accounts of two clocks to consider only two clocks. Finally, it disregards the temporal component that pertains to rods and the spatial component that pertains to clocks.

Still, if the elementary paradox is to be set aside as a gross over-simplification, there remains in its entirety the problem of working out a coherent account of the notion of measurement compatible with the complexity of Special Relativity. To this task we must now address our attention.

are invariant under permissible transformations, and so measurements valid in one reference frame are valid in all permissible frames.

On the suppositions of the Special Theory of Relativity, some revision is necessary. We shall consider how it affects 1) lengths of standard units, 2) lengths of measurable objects, 3) measurements, and 4) sizes.

First, a length results from fitting a size into a geometrical construction. On the Special Theory of Relativity, the relevant geometry is that of Minkowski space. The following characteristics of the lengths of standard units follow from the properties of this space or, what comes to the same thing, from the Lorentz-Einstein transformation.

1. In all inertial frames of reference a standard rod determines a four-dimensional interval of magnitude, unity. Similarly, in all inertial frames of reference a standard clock determines a four-dimensional interval of magnitude, ic, where i is the square root of minus one, and c is the velocity of light in vacuo.

2. A reference frame will be said to be normal to a standard rod, when the rod in the frame determines an interval with spatial component of magnitude, unity, and with temporal component of magnitude, zero.

Similarly, a reference frame will be said to be normal to a standard clock, when the clock in the frame determines an interval with a spatial component of magnitude, zero, and a temporal component of magnitude, unity.

3. Reference frames that are not normal to standard rods or standard clocks are in relative motion to

normal reference frames.

Inversely, in reference frames in relative motion to normal frames, standard rods determine the same invariant interval but now possess spatial components, H, and temporal components, $-Hu/c^2$ or Hu/c^2, according to the direction of the relative motion.

Similarly, in reference frames in relative motion to normal frames, standard clocks determine the same invariant interval, which, however, now possesses a spatial component, $-Hu$ or Hu, and a temporal component, H.

Secondly, there are to be determined the characteristics of the lengths of other measurable objects. Clearly, these lengths will have the same properties as the lengths of standard units. For both sets of lengths are subject to the same transformation equations.

Accordingly, for every measurable spatial object there is a group of normal reference frames, relatively at rest, and in them the object determines an interval with spatial component, A, and with temporal component, zero. In other reference frames in relative motion, the same object will determine an interval of the same magnitude but with spatial component, AH, and with temporal component, $-AHu/c^2$ or AHu/c^2, according to the direction of the relative motion.

Similarly, for every measurable temporal object, there is a group of normal reference frames, relatively at rest, and in them the object determines an interval with spatial component, zero, and with temporal component, B. In other inertial frames in relative motion, the same object will determine the same invariant interval, namely icB, but with a spatial component, -BHu or BHu, and with a temporal component, BH.

In the third place, measurements are to be considered, and they offer two distinct aspects.

For, in so far as measurements are numbers to be substituted into equations or to be derived by solving equations, they are identical with lengths. This follows from the nature of the coordinate system which, in the present case, deals only with measured lengths. Accordingly, all that has been said about lengths may now be repeated about measurements. A spatial magnitude will determine an invariant interval, A, with components, $[AH, -AHu/c^2]$, and a temporal magnitude will determine an invariant interval, icB, with components, $[BH, -BHu]$. In normal reference frames, H becomes unity, and u becomes zero, so that the components are $[A, 0]$ and $[0, B]$ respectively. Finally, in transformations to the left, the sign of u changes.

However, there is a further aspect to measurements. The numbers substituted into equations have to be derived from data, and the numbers derived from equations have to be verified in data. Thus, there arises the question whether Special Relativity modifies the concrete operation of measuring.

The general answer would seem to be that it does not. A measurement remains the number that stands to unity as the measurable object stands to a standard unit. However, within the frame-work of that general answer it will be well to advert to particular cases.

Ordinarily, simultaneity is determined in the same manner in selecting the point-instants at the ends of the standard unit and in selecting those at the ends of the measurable object. It will follow that spatial measurements ordinarily occur with the standard unit and the measurable object in the same reference frame and, since $A : 1 :: AH : H$, the result of measuring will be the number, A.

Still, this is not inevitable. Further, it may be fairly common to use a clock, stationary in a reference frame, to time a process that begins at one place in the frame and ends at another. Hence, besides the measurements that result when the object and the standard are taken in the same frame, namely, $A/1$, AH/H, $B/1$, BH/H, there are the measurements that result when they are in different frames. If one of these frames is normal, the results will be $AH/1$, A/H, $BH/1$, B/H: if neither frame is normal, one must distinguish two values of H, say H' and H'', so that the results may be AH'/H'', AH''/H', BH'/H'', BH''/H'. In other words, the actual process of measuring can involve the same ambiguities as are contained in the elementary paradox and, indeed, even more elaborate ambiguities.

Accordingly, we are brought to the conclusion that, while Special Relativity demands an operation of measuring that fundamentally is similar to measuring under Newtonian assumptions, still it adds new rules that either eliminate or correct some results which, on Newtonian assumptions, would be valid.

Chapter V: Space and Time. §4. Rods and Clocks.

Foot-note to p. ~~329~~ 277

The questions should be clarified. "Size" has been defined as an experiential conjugate that varies both from inner change in the object and from change of position of the observer. In the text I do not mean to deny perspectival variation of size. Similarly, I do not mean either to affirm ~~or~~ or to deny what I regard as meaningless, namely, that there is or is not an inner change of the object as referred to some absolute space. The question is whether an acceptance of special relativity logically entails any change in rods or clocks, and my answer is that no such change can be deduced. "Lengths" vary because reference frames vary; and reference frames vary because modes of determining simultaneity vary.

^x^ Foot-note on previous sheet

In the fourth place, there are the sizes of spatial and of temporal magnitudes. Do rods contract or expand? Do clocks run slow or fast? x Our answer will be negative, and our reasons run as follows:

First, it is difficult to suppose that rods and clocks should undergo such variation without a proportionate variation occurring in the objects that they measure: and if the proportionate variation occurs, then no explanation is provided for the relativity of lengths to reference frames.

Secondly, even if rods and clocks varied while other sizes do not vary, the required explanation would not be forthcoming. For rods and clocks and other sizes determine intervals that are invariant for all inertial reference frames. Moreover, these intervals exhibit temporal components for rods and other spatial magnitudes; and they exhibit spatial components for clocks and other temporal magnitudes. How does a contracting rod generate a temporal component? How does a decelerated clock generate a spatial component?

Thirdly, the evidence for contracting rods and decelerated clocks lies in the elementary paradox. Now we have no doubt that, on the suppositions of Special Relativity, it would be possible to reach such measurements as A/H, AH/1, B/H, BH/1, which are the lengthened and shortened rods and the faster and slower clocks. But the obvious explanation lies, not in any variation of the sizes of rods or clocks, but in the relativity of lengths and in the use of a standard unit in one reference frame to measure an object in another, significantly different, frame.

Fourthly, there is no aspect of the Special Theory of Relativity that is not accounted for by distinguishing between size and length, where length is constructed in accord with the geometry of Minkowski space. There follow immediately both the invariant intervals and the relativity of spatial and temporal components to reference frames. Moreover, this construction of length presupposes, not a variation in size, but a relatively of simultaneity. It was from a relative solution to the problem of synchronization that Special Relativity was evolved; and whenever such a solution is adopted, Special Relativity will follow even though no variation in size is admitted.

The point is worth illustrating. Suppose two planes flying in the same direction with the same constant velocity, so that the distance between them is constant. Let that distance be regarded as the standard unit, and suppose two observers, K and K', that determine simultaneity differently. Now consider the instant at which the first plane is at a point, P. Let us say that for K the second plane at the same instant is at some point, R. Then for K', since he determines simultaneity differently, the second plane must be at some nearer or further point, S, at the instant when the first plane is at P. Accordingly, though there is only one size, though this size is constant, though both observers agree that there is only one size and that it is constant, none the less, in virtue of different determinations of simultaneity, there are two lengths, PR and PS, and they are unequal with an inequality in some proportion to the relative velocities of the planes and the divergence between the two determinations of simultaneity.

While this illustration is, I believe, to the point, still it is only an illustration. One cannot take a relativity of simultaneity as postulate and from it deduce the Special Theory of Relativity. On the contrary, a relativity of simultaneity merely sets a problem; confronted with that problem, one adverts to the invariance of principles and laws; and it is by postulating the invariance of principles and laws under inertial transformations that one reaches the basic premise from which Special Relativity follows.

4.4 Summary

The aim has been to work out a general theory of measurement and thereby clarify the notions of measurable object, standard unit, measuring, and measurement peculiar to Special Relativity.

Measurement was seen to be the technique by which the scientist moves from the description of things as related to our senses to the explanation of things as related to one another.

Standard units were conceived as measurable objects that intrinsically stand on the same footing as other measurable objects but conventionally are given a unique status to simplify and systematize the formulation of the relations of things to one another.

The definitions of measurable objects of various kinds, the standardization of their respective units, the rules of measuring, and the nature of measurement were seen to depend on abstract presumptions and laws and, therefore, to be subject to revision along with revisions of the presumptions and the laws.

This generic notion of measurement was then applied to measurements of spatial and temporal magnitudes.

A basic distinction was drawn between the experiential conjugate, size, and the pure conjugate, length. The former is correlative to our experience. The latter is implicit in a geometrical structure of definitions, postulates, and inferences.

The transition from Newtonian to Einsteinian physics is a transition from length, as implicit in Euclidean geometry, to length, as implicit in Minkowski space. It drops invariant spatial and temporal lengths. It introduces invariant four-dimensional intervals with variable spatial and temporal components. While it grants no special significance to reference frames at rest, still it does imply a position of privilege for normal reference frames, in which spatial magnitudes have a zero temporal component and temporal magnitudes have a zero spatial component. Thus, an interval, A, which is a real number, has the components $\left[\underline{A}H, -\underline{A}Hu/c^2\right]$ which become $\left[\underline{A}, 0\right]$ in a normal reference frame; and an interval, $\underline{icB}$, which is an imaginary number, has the components $\left[-\underline{B}Hu, \underline{B}H\right]$, which become $\left[0, \underline{B}\right]$ in a normal reference frame. It is to be noted that the distinction between the spatial and the temporal is as sharp as the distinction between real and imaginary numbers, that the lengths of standard units are but particular cases of the lengths of other measurable objects, that the transformation properties of unit and of other lengths are the same, that in a Minkowski manifold lengths are already measured so that measurements are coincident with lengths, that in the operation of measuring, there arise in Special Relativity ambiguities that do not exist and so do not have to be solved on Newtonian suppositions.

However, while Special Relativity involves a revision of the notions of lengths and of measurements and while it introduces a new caution in the operation of measuring, it does not imply the expansion or contraction of rods or the acceleration or deceleration of clocks. In other words, the unit divisions of the axes in the coordinate systems are constituted, not by the size, but by the length of standard distances and standard durations. Such lengths are relative to reference frames, but this relativity of length arises, not from change of size, but from the inter-dependence of determinations of length and of simultaneity. What corresponds to change of size is, not a mere transformation of reference frames, but a variation in the intervals, A or icB. A variation in some of these intervals corresponds to a variation in some measurable objects: a proportionate variation in all of these intervals suggests that the standardization of units needs to be corrected and revised.

Might I suggest that, on this showing, there vanishes the arbitrary division of the world of physics into rods and clocks and, on the other hand, all other objects? Such arbitrariness is noted and regretted by Prof. Einstein in his Autobiography. (Albert Einstein, Philosopher-Scientist, ed. P.A.Schilpp, The Library of Living Philosophers, New York, 1949 and 1951, p. 59)

Schilpp

It would seem to vanish 1) inasmuch as physics is set the task of assigning invariantly expressed abstract relations to account not only for experienced colors and sounds but equally for experienced extensions and durations, 2) inasmuch as these relations are reached by formulating and verifying hypotheses, 3) inasmuch as notions of length and measurement and the standardization of units form internal parts of the hypothesis to be verified, 4) inasmuch as the hypothesis assigns the same properties to lengths of standard units as to lengths of other measurable objects, and 5) inasmuch as frames of reference have their units constituted, not by the sizes of rods and clocks, but by their theoretically defined lengths.

Finally, it would seem that the foregoing account of rods and clocks in Special Relativity might easily be adapted to the requirements of General Relativity. In General Relativity there remains the invariant four-dimensional interval; there remain its spatial and its temporal components; there remains the covariance of these components in different reference frames. The basic differences are that the components now are curvilinear and that specifications of coordinates are not virtual measurements of distance or duration.

On the other hand, it is not to be claimed that our account of measuring is completely general. Rather that distinction seems to pertain to Quantum Theory viewed as a theory of measurements. For if it is true that all measuring is abstractive both in the sense that it replaces sets of data by series of approximate numbers and in the sense that it relates the numbers not to our senses but to one another, still the relations may be systematic or non-systematic; and non-systematic relations, no matter what their origin, can be manipulated theoretically only in a context that envisages statistical laws.

human intellect is a potential omnipotence, a _potens omnia facere et fieri_. But Aquinas could exploit that affirmation in a manner that would have startled Aristotle.

First, he recognized an unrestricted desire to know. As soon as we learn of God's existence, we wish to understand His nature. To achieve such understanding is beyond the power of our natural capacity, yet in such achievement lies our spontaneously desired beatitude. (I, 12, 1, ff.; I-II, 3, 8; 5, 5).

Secondly, the unrestrictedness native to intellect grounds the affirmation that the object of intellect has to be being. Because intellect is _potens omnia fieri_, its object is _ens_. (I, 79, 7, c). Being and everything are equivalent notions.

Thirdly, for the same reason, an intellect fully in act must be infinite and uncreated act. Any created intellect must in some manner be potential, and our intellects start from a zero of potentiality. (I, 79, 2, c. _CG_ II, 98).

Fourthly, none the less, being is _per se_ and naturally known to us (_CG_ II, 83#31), and it cannot be unknown to us. (_De Ver._ 11, 1, 3m). Avicenna had interpreted Aristotle's agent intellect as some separate immaterial substance. Aquinas found it immanent within us; the light of intelligence, which is in us, performs the functions Aristotle attributed to agent intellect, and, moreover, Aristotle compared agent intellect to a light. (_CG_ II, 77#5). Augustine had advanced that our knowledge of truth originated, not without but within us, yet not simply within us, but in some illumination in which we consulted the eternal

No!

Chapter XV : Elements of Metaphysics. Foot-note to p.704.

The relation of potency, form, and act, as defined, to Scholastic potentia, forma, actus, may be bracketed under the three headings of technique, principle, and method.

My definition are systematic. In contract, the normative influence exercised by the Scholastic disputation set a premium on definitions that were nominal, that prescinded from systematic views, that stated what would be meant by the members of any school.

Secondly, the principles on which my definitions are based would be rejected by the conceptualist wing of Scholastic thinkers. Because conceptualists deny insight, they eliminate what I mean by form. Because they conceive abstraction as impoverishing, they eliminate the distinction I draw between potency and form and deny its implication that matter is a principle of individuation. Because they consider judgment to be an adhaesio mentis that does not augment the content of knowledge, they eliminate the distinctio I draw between form and act and deny its implication of a real distinction between essence and contingent existence.

Thirdly, the Scholastics that employ systematic definitions and hold principles similar to my own follow quite a different method. While the present metaphysics is epistemologically constructed in terms of the causa cognoscendi, theirs is ontologically constructed in terms of the causa essendi ; and while my starting-point is restricted to proportionate being, theirs contains an explicit reference to transcendent being in a theorem on the intrinsic illimitation of act and the limiting roles of form and potency. Accordingly, it is only at the end of Chapter XIX that the isomorphism between Thomism and the present metaphysics can begin to appear.

II Diagram

For the complete, draw a diagram and mark clearly the known & the unknown quantities.