Insight.

Chapter II.

Heuristic Structures of Empirical Method.

In the previous chapter insight was examined in a static fashion. It was related to inquiry, to images, to empirical data, and to different types of positive and negative explanatory concepts. But if a set of fundamental notions has been introduced, no effort has been made to capture the essential dynamism of human intelligence. Now a first move must be made in this direction and, as empirical science is conspicuously and methodically dynamic, it will be well to begin by outlining the similarities and dissimilarities of mathematical and scientific insights.

1.1 Similarities of Mathematical and Scientific Insights.

Galileo's determination of the law of falling bodies not only is a model of scientific procedure but also offers the attraction of possessing many notable similarities to the already examined process from the image of a cart-wheel to the definition of the circle.

In the first place, the inquiry was restricted to the immanent intelligibility of a free fall. Just as we ruled out of consideration the purpose of cart-wheels, the materials from which they are made, the wheelwrights that make them, and the tools that wheelwrights use, so also

then follows pp 24-28 (early manuscript)

1. Mathematical and Scientific Insights.

2. ~~XX~~ So far our illustrations of insight have been drawn from the field of mathematics. There have been examined the definition of the circle, the transition from arithmetic to algebra, the distinction between different kinds of infinite ~~sets. However, our basic illustration initial example was not mathematical but mechanical. It was Archimedes' discovery of the principles of displacement and of specific gravity.~~ sets. It is true that we began from the story of Archimedes' discovery of principles of displacement and specific gravity. But then we were content merely to indicate the more obvious features of insight and made no attempt to analyse the precise nature of the origin and development of scientific knowledge. Such an analysis must now be tackled.

Similarities.

~~XXI~~ 1.1 Galileo's determination of the law of falling bodies not only is a model of scientific procedure but also offers the attraction of possessing many notable similarities to the already examined process from the image of the cart-wheel to the definition of the circle.

In the first place, the inquiry was restricted to the immanent intelligibility of a free fall. Just as we ruled out of consideration the purpose of cart-wheels, the materials from which they are made, the wheelwrights that make them, and the tools that wheelwrights use, so also Galileo was uninterested in the final cause of falling, he drew no distinction between the different materials that fall, he made no effort to determine what agencies produce a fall.

Secondly, just as we started from a clue, the equality of the spokes, so too Galileo supposed that some correlation was to be found between the measurable aspects of falling bodies. Indeed, he began by showing the error in the ancient, Aristotelian correlation that bodies fell according to their weight. Then he turned his attention to two measurable aspects immanent in every fall: the body traverses a determinate distance; it does so in a determinate interval of time. By a series of experiments he provided himself with the requisite data and obtained the desired measurements. ~~From the measurements to the formula~~ Then, he discovered that the measurements would satisfy a general rule: the distance traversed is proportional to the time squared. It is a correlation that has been verified directly and indirectly for over four centuries.

something similar

Thirdly, once we had defined the circle, we found ourselves in a realm of the non-imaginable, of the merely supposed. Strangely, ~~the same thing~~ happens when one formulates the law of falling bodies. It holds in a vacuum, and to realize a perfect vacuum is impossible. What can be established experimentally is that the more closely ~~we~~ one/ approximates to the conditions of a vacuum, the more accurate the law of constant acceleration is found to be.

~~In the fourth place, however, there is a profound difference between the definition of the circle and the law of falling bodies. The insight that grounds the geometrical definition is a grasp of necessity and impossibility: if the radii are equal, the curve must be round; if the radii are unequal, the curve cannot be round. But the insight that grounds the formulation of the law involves no grasp of necessity or impossibility. However well the law is understood,~~

Dissimilarities.

1.2 ~~2x2~~ But besides similarities, there also are differences and these are perhaps more instructive.

In reaching the definition of the circle, it ~~is~~ was sufficient to take as our starting-point the mere image of a cart-wheel. There was no need for field-work. But to reach the law of falling bodies Galileo had to experiment. Climbing the tower of Pisa and constructing inclined planes were an essential part of his job, for he was out to understand, not how bodies are imagined to fall, but how in fact they fall.

Secondly, the data that give rise to insight into roundness are continuous, but the data that give rise to insight into the law of falling bodies are discontinuous. One can imagine the whole cart-wheel or a whole loop of very fine wire. But no matter how many experiments on falling bodies one makes, all one can obtain is a series of separate points plotted on a distance-time graph. No doubt, it is possible to join the plotted points by a smooth curve, but the curve ~~is~~ represents, not data that are known, but a presumption of what understanding will grasp.

Thirdly, the insight into the image of the wheel grasps necessity and impossibility: if the radii are equal, the curve must be round; if the radii drawn from the center/are unequal, the curve cannot be round. But the insight into the discontinuous series of points on the graph consists in a grasp, not of necessity or impossibility, but simply of possibility. The simplest smooth curve could represent the law of falling bodies. But any of a vast range of more elaborate curves could equally well pass through all the known points.

Fourthly, once one catches on to the law of the circle, the insight and consequent definition exert a backward influence upon imagination. The geometer imagines dots but thinks of points; he imagines fine threads, but thinks of lines. The thinking is exact and precise, and imagination does its best to keep pace. In like manner the empirical investigator will ~~tend to give his images endow his images the closest possible~~ tend to endow his images with the closest possible approximation to the laws he conceives. But while his imagination will do its ~~best, the sensible data will make no effort at~~ best, while his perceptions will be profoundly influenced by the habits of his imagination, none the less the data that are available for the ideal observer make no effort towards such ~~for~~ conformity. They go their own way with their unanalysed multiplicity and their refractoriness to measurements that are more than approximate.

Fifthly, as we have seen, higher viewpoints in mathematics are reached inasmuch as initial images yield insights, ~~and definitions~~ insights yield definitions and postulates, definitions and postulates guide symbolic operations, and symbolic operations provide a more general image ~~in which~~ in which the insights of the higher viewpoint are emergent. Now in empirical method there is a similar circle but it follows a slightly different route. The operations that follow upon the ~~definition of laws~~ formulation of laws are not merely symbolic. For the formulation expresses a grasp of possibility. It is a hypothesis. It provides a basis for deductions and calculations no less than mathematical premises. But it also provides a ~~premise~~ basis for further observations and experiments. It is such observation and experimentation, directed by a hypothesis, that sooner or later turns attention to data that initially were overlooked or

Chapter II: Heuristic Structures.

Foot-note to page 80.

Insert asterisk in text at end of first paragraph, line 13.

* Because insight is into the presentations of sense or the representations of imagination, the third step in the solution of such problems is facilitated by drawing a diagram and marking all relevant quantities. In the present instance the equation becomes evident on inspection when one has marked the three distances, x, $x/12$, and 15.

neglected; it is attention to such further data that forces the revision of initial viewpoints and effects the development of empirical science.

The circuit, then, of mathematical development may be named immanent: it moves from images through insights and conceptions to the production of symbolic images whence higher insights arise. But the circuit of scientific development includes action upon external things: it moves from observation and experiment to tabulations and graphs, from these to insights and formulations, from formulations to forecasts, from forecasts to operations, in which it obtains fresh evidence either for the confirmation or for the revision of existing views.

2. Classical Heuristic Structures

In one respect this brief sketch must be completed at once. Quite airily, we have spoken of the initial clue. But just what is it? Where does it come from? Is it mere guess-work? One can be led on quite naturally to the definition of the circle, if one begins from a suspicion that a cart-wheel is round because its spokes are equal. Similarly, one can proceed in intelligible fashion to the determination of the law of falling bodies, provided one presumes initially that the law will be a correlation of measurable aspects of a free fall. But this only makes the origin of the clue or hint or suggestion or presumption all the more significant.

2.1 An Illustration from Algebra

With another bow, then, to Descartes' insistence on understanding extremely simple things, let us examine the algebraist's peculiar habit of solving problems by announcing: Let x be the required number.

Thus, suppose that the problem is to determine when first after three o'clock the minute hand exactly covers the hour hand. Then, one writes down, Let x be the number of minutes after three o'clock. Secondly, one infers that while the minute hand moves over x minutes, the hour hand moves over x/12 minutes. Thirdly, one observes that at three o'clock the hour hand has a 15 minute start. Hence,

$$x = x/12 + 15 = 16\ 4/11$$

The procedure consists in 1) giving the unknown a name or symbol, 2) inferring the properties and relations of the unknown, 3) grasping the possibility of combining these properties and relations to form an equation, and 4) solving the equation.

2.2 "Nature"

Now let us generalize.

In every empirical inquiry there are knowns and unknowns. But the knowns are apprehended whether or not one understands; they are the data of sense. The unknowns, on the other hand, are what one will grasp by insight and formulate in conceptions and suppositions.

Accordingly, let us bestow a name upon the unknown. Rather, let us advert to the fact that already it has been named. For what is to be known by understanding these data is called their nature. Just as in algebra the unknown number is x, until one finds out what the number is, so too in empirical inquiry the unknown to be reached by insight is named "the nature of..." Once Galileo discovered his law, he knew that the nature of a free fall was a constant acceleration. But before he discovered the law, from the mere fact that he inquired, he knew that a free fall possessed a nature, though he did not know what that nature was.

The first step in the generalization is, then, that just as the mathematician begins by saying, Let the required number be x, so too the empirical inquirer begins by saying, Let the unknown be the nature of...

2.3 Classification and Correlation. ~~XXXXX~~ Next, similars are similarly understood.

Hence, because individuality pertains to the empirical residue, one knows at once that the "nature of..." will be universal, that when one understands these data, then one will understand similar data in exactly the same fashion.

Accordingly, just as the mathematician follows up his naming of the unknown as x by writing down properties of x, so too the empirical inquirer follows up his declaration that he seeks the "nature of..." by noting that that "nature of..." must be the same for all similar sets of data.

But similarities are of two kinds.

There are the similarities of things in their relations to us. Thus, they may be similar in color or shape, similar in the sounds they emit, similar in taste or odor, similar in the tactile qualities of the hot and cold, wet and dry, heavy and light, rough and smooth, hard and soft.

There also are the similarities of things in their relations to one another. Thus, they may be found together or apart. They may increase or decrease concomitantly. They may have similar antecedents or consequents. They may be similar in their proportions to one another, and such proportions may form series of relationships, such as exist between the elements in the periodic table of chemistry or between the successive forms of life in theory of evolution.

Now sensible similarities, which occur in the relations of things to our senses, may be known before the "nature of..." has been discovered. They form the basis of preliminary classifications. They specify the "nature of...," so that one states that one is seeking the nature of color, the nature of heat, the nature of change, the nature of life.

On the other hand, similarities that reside in the relations of things to one another are the proximate materials of insight into nature. Hence, the empirical inquirer, to emphasize this fact, will say that his objective is not merely the "nature of..." but more precisely the unspecified correlation to be specified, the undetermined function to be determined.

The second step in the generalization is, then, that just as the mathematician states that he seeks an x which has such and such properties, so too the empirical inquirer states that he seeks a "nature of..." where the nature antecedently is specified by a classification based on sensible similarity and consequently will be known when some indeterminate function is determined.

The reader will observe that Galileo differed from his Aristotelian opponents by taking this second step. The Aristotelians were content to talk~~x~~ about the nature of light, the nature of heat, etc. Galileo inaugurated modern science by insisting that the nature of~~x~~ weight was not enough;

from sensible similarity, which resides in the relations of things to our senses, one must proceed to relations that hold directly between things themselves.

2.4 Differential Equations.

Now the correlations and functions that relate things directly to one another are determined empirically by measuring, plotting measurements on graphs, and grasping in the scattered points the possibility of a smooth curve, a law, a formulation. But our present concern is with the antecedent, heuristic clues. Accordingly, we recall that, besides individuality, the continuum also pertains to the empirical residue and, as well, that just as the universal is reached by abstracting from the individual, so also the techniques of the infinitesimal calculus deal with the intelligibility reached by abstracting from the non-countable infinity of the continuum.

The third step, then, in our generalization is the observation that, where the mathematician says let x be the required number, ~~so~~ the empirical inquirer can say let some indeterminate function, $f(x, y, z, ...) = 0$, be the required function. Further, just as the mathematician reaches x by making statements about it, so too the empirical inquirer can move towards the determination of his indeterminate function by writing down the differential equations which it must satisfy.

This procedure is named by Lindsay and Margenau in their Foundations of Physics the "Method of Elementary Abstraction." They illustrate it by ~~examining~~ examining the general features of a fluid in motion. Thus, if the fluid is continuous, then at every point in the fluid there will be the velocity components, u, v, w, and a density, r. If the fluid is not vanishing into gas, then the excess rate of out-flow over in-flow with respect to any infinitesimal volume will equal the rate of decrease of density in that volume. Hence, ~~one has~~ there may be derived the equation,

$$\partial(ru)/\partial x + \partial(rv)/\partial y + \partial(rw)/\partial z = -\partial r/\partial t$$

Further, if the motion is only in one direction, two of the terms on the left-hand side vanish. If the fluid is incompressible so that the density does not vary in time, the ~~right-hand~~ term on the right-hand side becomes zero. If the fluid is also ~~incompressible~~ homogeneous, so that the density does not vary in space, then the density, r, vanishes from the expressions on the left-hand side. Finally, if the velocity components, u, v, w, are equal to the first partial derivatives of some function of the coordinates, x, y, z, there arises Laplace's equation.

The foregoing equation of continuity can be combined with other equations based on similarly general considerations. Thus, by shifting from velocity and density to acceleration and pressure, three further differential equations can be obtained. By adding ~~suitable~~ suitable assumptions and restrictions, there can be worked out the differential equation of a wave motion (See Lindsay and Margenau, pp. 29 ff).

What is happening? Consider the algebraic procedure that we are generalizing and observe the isomorphism. Where before we said, Let x be the required number, now we say, Let $f(x, y, z, t) = 0$ be the required function. Where before we noted that, while the minute hand moves over x minutes, the hour hand moves over $x/12$ minutes, now we work out a differential

the follow pp 85-107

What is happening? Consider the algebraic procedure that we are generalising and observe the isomorphism. Where before we said, let x be the required number, now we say, Let the function, $f(x, y, z, t) = 0$, be the required correlation. Where before we noted that, while the minute hand moves over x minutes, the hour hand moves over $x/12$ minutes, now we work out a differential equation that expresses mathematically certain very general features of the data. Where before we appealed to the fact that at three o'clock the hour hand had a fifteen minute start, now we turn our attention to the boundary conditions that restrict the range of functions satisfying the differential equation.

2.5 Invariance.

Though a less inadequate account of the notion of invariance will be attempted in examining the notions of Space and Time in Chapter V, at least some mention of it should be made in the present outline of scientific clues and anticipations. Accordingly, we recall that the differences of particular places and particular times pertain to the empirical residue and, for that reason, not only are scientific discoveries independent of the place and time of their origin but also they can claim to be equally and uniformly valid irrespective of merely spatio-temporal differences. Hence, for example, the formulae for chemical compounds not only have the same intelligibility and meaning but also exactly the same symbolic representation no matter what the place or time. However, physical principles and laws are involved in a difficulty. For they regard motions of one kind or another; motions are changes in place and time; places and times lead to reference frames constructed to include and designate all points and instants relatively to a particular origin and orientation. It follows that if physical principles and laws refer to motions, they also refer to the particular origin and orientation of some particular reference frame and, unless a special effort is made, change in the choice of reference frame may result in change in the statement of the principle or law. On the other hand, when a special effort is made, the mathematical expression of physical principles and laws undergoes no change in form despite changes in spatio-temporal standpoint and then the mathematical expression is said to be invariant under some specified group of transformations.

Briefly, then, the meaning of invariance is that 1) all scientists expect their correlations and laws to be independent of merely spatio-temporal differences, 2) physicists are confronted with a special difficulty inasmuch as they have to use reference frames, and 3) physicists surmount their peculiar difficulty by expressing their principles and laws in mathematical equations that remain invariant under transformations of frames of reference.

However, to determine under which group of transformations invariance is to be achieved, some further principle has to be invoked and, in fact, in different scientific theories different principles are invoked. Of these the most general is the principle of equivalence which asserts that physical principles and laws are the same for all observers. Now at first sight this statement seems ambiguous. Does it mean that physical objects look the same from all observational standpoints? Or does it mean that physical principles and laws are simply and completely outside the range of seeing, hearing, touching, feeling, and all other direct and indirect acts of observing?

While some writers seem to favor the former view, there can be little doubt about Einstein's position. Moreover, that position follows quite plausibly from the premise that empirical science seeks not the relations of things to our senses but their relations to one another. For, as has been remarked, observations give way to measurements; measurements relate things to one another rather than to our senses; and it is only the more remote relations of measurements to one another that lead to empirical correlations, functions, laws. Now clearly if laws are reached by eliminating the relations of things to the senses of observers and by arriving at relations between the measured relations of things to one another, then there exists an extremely solid foundation for the affirmation that principles and laws are the same for all observers because they lie simply and completely outside the range of observational activities. It is, for example, not the appearance of colors but the general explanation in terms of wave-lengths of light that is exactly the same no matter what may be the state of observers' eyes, the lighting by which they see, or the speed with which they may happen to be in relative motion.

Hence, if physical principles and laws are independent of any movement of observers, they should be equally independent of any similar movement of reference frames. But observers may be moving with any linear or angular velocity provided the motion is continuous and provided it involves no excursions into the ~~[illegible]~~ imaginary sections of a manifold constructed by introducing complex numbers. It follows that physical principles and laws should be independent of similar movements of reference frames. Accordingly, by the principle of equivalence the mathematical expression of physical principles and laws is to be expected to be invariant as long as transformation equations are continuous functions of real variables.

To implement this conclusion, which is no more than a general anticipation based on cognitional theory, two further steps are required. First, the broad invariance that we have described has to be conceived precisely in terms of tensors. Secondly, appropriate empirical hypotheses have to be formulated and verified. But by these steps there are reached the General Theory of Relativity and the Generalized Theory of Gravitation and it may not be amiss to note that our remote anticipation offers a simple explanation for certain aspects of those theories. For what was anticipated was a non-relatedness of abstract laws to observers. It follows that the consequences of the anticipation should not be verified 1) if the laws lose their abstract character through particularization (*), or 2) in if investigation concentrates on the frequencies of concrete events accessible to observers as seems to be the case in Quantum Mechanics.

(*) See Lindsay and Margenau, p. 368.

A less general anticipation of invariance is contained in the basic postulate of Special Relativity. Already in illustrating inverse insight we have had occasion to put this postulate in the form of an explanatory syllogism in which the major premise expressed an anticipation of invariance and the minor premise enounced the defect of intelligibility in inertial transformations. On the present analysis, then, the difference between the anticipations represented respectively by General and by Special Relativity is that, while both expect invariant mathematical expression to result from the abstractness of principles and laws, General Relativity implements this expectation by invoking a direct insight into the significance of measurements but Special

Relativity implements it by invoking an inverse insight into the insignificance of constant velocity.

The exact nature of this difference may be clarified by two further remarks. On the one hand, it does not prevent Special Relativity from being regarded as a particular case of General Relativity, for General Relativity does not attribute any significance to constant velocity, and Special Relativity primarily regards laws reached by relating measurements to one another. On the other hand, the difference is a difference not merely in degree but also in kind, for the anticipations of General Relativity do not hold when the results of investigations include relations to observers, but the anticipations of Special Relativity do hold as long as the insignificance of constant velocity is extended to the whole of physics. So perhaps one may explain the fact that the anticipations of Special Relativity have been meted successfully with Quantum Mechanics (*).

(*) See Lindsay and Margenau, pp. 501 ff.

A third and still less general anticipation of invariance has been attributed retrospectively to Newtonian dynamics, and it is not difficult to ~~grasp~~ grasp in terms of insight the justice of this view. For, as has been noted, the defect in intelligibility known in inverse insight is formulated only by employing a positive context of concomitant direct insights. In particular, it has been remarked that the defect of intelligibility in constant velocity was expressed for mechanics by Newton in his first law of motion but for physics generally by Einstein in the basic postulate of Special Relativity. Accordingly, one can move backwards from Einstein to Newton if 1) one holds fast to the defective intelligibility in constant velocity and 2) one changes the concomitant context of direct insights in terms of which the inverse insight regarding constant velocity is expressed.

Now the relevant differences in the concomitant context are threefold. First, Special Relativity regards all physical principles and laws, but Newtonian dynamics is concerned primarily with mechanics. Secondly, Special Relativity is primarily a field theory, that is, it is concerned not with the efficient, instrumental, material, or final causes of events, but with the intelligibility immanent in data; but Newtonian dynamics seems primarily a theory of efficient causes, of forces, their action, and the reaction evoked by action. Thirdly, Special ~~x~~ Relativity is stated as a methodological doctrine that regards the mathematical expression of physical principles and laws, but Newtonian dynamics is stated as a doctrine about the objects subject to laws.

From these differences it follows that what Einstein stated for physics in terms of the transformation properties of the mathematical expression of principles and laws, Newton stated for mechanics in terms of the forces that move bodies. In both cases what is stated is a negation of intelligibility in constant velocity. But the Einsteinian context makes the statement an affirmation of invariance despite inertial transformations, while the Newtonian context makes the statement an affirmation of continued uniform motion in a straight line despite the absence of external forces. Finally, as the Einsteinian statement may be regarded as a methodological rule governing the expression of physical principles and laws, so the Newtonian statement may be regarded as a general boundary condition complementing the laws that equate 1) force with change of momentum and 2) action with an equal and opposite reaction.

2.6 Summary.

Our concern has been the methodical genesis of insight. Scientists achieve understanding, but they do so only at the end of an inquiry. Moreover, their inquiry is methodical, and method consists in ordering means to achieve an end. But how can means be ordered to an end when the end is knowledge and the knowledge is not yet acquired? The answer to this puzzle is the heuristic structure. Name the unknown. Work out its properties. Use the properties to direct, order, guide the inquiry.

In prescientific thought what is to be known inasmuch as understanding is achieved is named the "nature of ..." Because similars are understood similarly, the "nature of ..." is expected to be the same for all similar data, and so it is specified as the nature of light, the nature of heat, and so forth, by constructing classifications based on sensible similarity.

Scientific thought involves a more exact anticipation. What is to be known inasmuch as data are understood is some correlation or function that states universally the relations of things not to our senses but to one another. Hence, the scientific anticipation is of some unspecified correlation to be specified, some indeterminate function to be determined; and now the task of specifying or determining is carried out by measuring, by tabulating measurements, by reaching an insight into the tabulated measurements, and by expressing that insight through some general correlation or function that, if verified, will define a limit on which converge the relations between all subsequent appropriate measurements.

This basic anticipation and procedure may be ~~entt~~ enriched in two further manners. First, functions are solutions of differential equations; but in many cases relevant differential equations can be deduced from very general considerations. Hence, the scientist may anticipate that the function, which is the object of his inquiry, will be one of the solutions of the relevant differential equations. Secondly, the functions that become known in the measure that understanding is achieved are, both in origin and in application, independent of the differences of particular places and particular times. In such a science of physics this anticipation of independence becomes formulated as the invariance of principles and laws under groups of transformations, and different ~~main~~ grounds are invoked to determine which group of transformation is to leave the mathematical expression of laws unchanged in form. So a direct insight into the significance of measurements yields the anticipations of General Relativity; an inverse insight into the insignificance of constant velocity yields the anticipations of Special Relativity; and a restriction of this inverse insight to the context of Newtonian dynamics yields the anticipations that sometimes are named Newtonian relativity.

Such in brief are the anticipations constitutive of classical heuristic structure. The structure is named classical because it is restricted to insights of a type most easily identified by mentioning the names of Galileo, Newton, Clerk-Maxwell, and Einstein. It is named heuristic because it anticipates insights of that type and, while prescinding from their as yet unknown contents, works out their general properties to give ~~now~~ methodical guidance to investigations. It is named a structure because, though operative, it is not known explicitly until oversight of insight gives way to insight into insight.

In particular one should observe that classical heuristic structure has no suppositions except the minimal suppositions that insights of a certain type occur and that ~~inqu-~~ inquiry aiming at such insights may be not haphazard but methodical. Further, advertence to classical heuristic structure has no additional suppositions except the possibility of an insight that grasps the set of relations linking methodical inquiry with anticipated insights, data, similarities in data, measurements, curve-fitting, indeterminate functions, differential equations, the principle of inertia, Special Relativity, and General Relativity. If there has been communicated some grasp of such diverse objects within the unity of a single view, then there has been communicated an insight into the genesis of insight. No doubt, that is a very small thing. An insight is no more than an act of understanding. It may prove to be true or false or to hold some intermediate position of greater or less probability. Still it is solely the communication of that act of understanding that has been our aim and, if the reader has been concerned with ~~anything~~ anything else, he has done all that is necessary to miss the little we have had to offer in the present context.

A further observation is not without its importance. Precisely because our suppositions and our objective have been so restricted, our account of classical heuristic structure is essentially free from any opinion about corpuscles, waves, causality, mechanism, determinism, the uniformity of nature, truth, objectivity, appearance, reality. It follows immediately that if we venture to use the name, "classical," we use it without being involved in any of the extra-scientific views that historically have been associated with scientific discoveries and, to a greater or less extent, have influenced their interpretations. This point is, of course, of considerable importance at a time when a new statistical heuristic structure has grown enormously in prestige and it has become a matter of some obscurity whether the new approach conflicts with the assumptions of earlier science or merely with the extra-scientific opinions of earlier scientists. Finally, if we may close this section on a still more general note, it is not perhaps rash to claim that an analysis of scientific procedures in terms of insight is also new and that the value of such analysis cannot be tested except by working out its implications and confronting them, not with opinions on science based on other analyses, but solely with strictly scientific anticipations, procedures, and results.

3. Concrete Inferences from Classical Laws.

Before advancing to a consideration of statistical heuristic structure, it will be well to ask just how far the full realization of classical anticipations would bring the scientist towards an adequate understanding of data. Accordingly, we ask about the range of concrete inferences from classical laws and we do so all the more readily because discussions of this topic seem to have suffered from an oversight of insight.

For just as insight is a necessary intermediary between sets of measurements and the formulation of laws, so also it is needed in the reverse process that applies known laws to concrete situations. ~~Accordingly~~ Hence, a concrete ~~new~~ scientific inference has not two but three conditions: it supposes information on some concrete situation; it supposes knowledge of laws; and it supposes an insight into the given situation. For it is only by the insight that one can know 1) which laws are to be selected for the inference, 2) how the selected laws are to be combined to represent the spatial and dynamic configuration of the concrete situation, and 3) what dimensions in the situation are to be measured to supply numerical values that particularise the selected and combined laws.

Further, such inferences can be carried out in two manners. While practical people wait for concrete situations to arise before attempting to work out their consequences, theoretical minds are given to anticipating ideal or typical cases and to determining how a deduction could be carried out in each case.

Now in these anticipatory concrete inferences a different type of insight comes into play. For in the practical inference the situation~~s~~ determines the relevant insight and the insight determines the selection, combination, and particularization of laws. But in the anticipatory inference insight is creative and constructive. It is not hampered by any given situation. Rather it tends to be a free ~~t~~ exploration of the potentialities of known laws, and its principal fruit is the formulation of ideal or typical processes that are dominated throughout by human intelligence. For in such processes the basic situation is any situation that satisfies the requirements of the constructive insight and, provided the process is closed off against all extraneous influence, every antecedent and consequent situation must assume the dimensions determined by the successive stages of the imaginative model.

Moreover, it can happen that such ideal or typical processes can be verified in a sequence of concrete situations, and then three very notable consequences follow. In the first place, some insight or some set of unified insights can grasp not only the process as a whole but also every event in the whole. Secondly, this single insight or single unified set can be expressed in a corresponding combination of selected laws and any situation can be deduced from any other without any explicit consideration of intervening situations. Thirdly, when such processes exist and their laws are as yet unknown, their investigation enjoys a number of singular advantages. For the intelligible unity of the whole process implies 1) that data on any situation are equivalent to data on the whole process, 2) that if data are found to be significant in any situation, then similar data will be significant in every other situation, and 3) that the accuracy ~~am~~ of reports on any situation can be checked by inferences from reports on other situations. Moreover, once initial difficulties are overcome and basic insights are reached, the investigation approaches a supreme moment when all data suddenly fall into a single perspective, sweeping yet accurate deductions become possible, and subsequent exact predictions regularly will prove to have been correct.

However, if the nature of statistical inquiry is to be understood, it is of considerable importance to grasp that a quite different type of process not only can be constructed but also probably can be verified. Accordingly, let us divide ideally constructed processes into systematic and non-systematic. Let us define systematic processes by the already enumerated properties that, other things being equal, 1) the whole of a systematic process and its every event possess but a single intelligibility that corresponds to a single insight or single set of unified insights, 2) any situation can be deduced from any other without an explicit consideration of intervening situations, and 3) the empirical investigation of such processes is marked not only by a notable facility in ~~the~~ ascertaining and checking abundant and significant data but also by a supreme moment when all data fall into a single perspective, sweeping deductions become possible, and subsequent exact predictions regularly are fulfilled.

Now whenever a group or series is constructed on determinate principles, it is always possible to construct a different group or series by the simple expedient of violating the determinate principles. But the group of systematic processes is constructed on determinate principles. Therefore, by violating the principles one can construct other processes that are non-systematic.

It is to be noted that the construction of non-systematic processes rests on the same knowledge of laws and the same creative intelligence as the construction of systematic processes. Hence if one inclines to enlarge the group of systematic processes by postulating full knowledge of laws and an unlimited inventiveness, one must grant that the group of non-systematic processes also is constructed from an equally full knowledge of laws and an equally unlimited (though perhaps perverse) inventiveness Finally, though we do not know all laws, none the less we can form the general notion of the systematic process; and similarly despite our ignorance of many laws we also can form the general notion of the non-systematic process.

For, in the first place, if non-systematic process is understood, the understanding will be multiple. There will be no single insight, or single set of unified insights, that masters at once the whole process and all its events. The only correct understanding will be either a set of different insights or else a set of different unified sets. In the former case the different insights will not be unified intelligibly and so they will not be related to one another in any orderly series or progression or grouping whatever. In the latter case the different sets of unified insights will have no higher intelligible unity and so they will not be related to one another in any orderly series or progression or grouping whatever. Finally, let us say that a series, progression, grouping is orderly if the relations between the elements of the series, progression grouping either 1) can be grasped by an insight that can be expressed in general terms or 2) can be concluded from any single insight or any single set of unified insights.

Secondly, because different parts of the process are understood differently, there can be no single combination of selected laws that holds for the whole process. On the contrary, for every different insight or different set of unified insights there will be a different combination and perhaps even a different selection of laws. Again, just as the different insights or unified sets of insights, so the different selections and combinations will not satisfy any orderly series or progression or grouping whatever.

Thirdly, such non-systematic process may be deducible in all its events. Let us suppose 1) the absence of extraneous interference, 2) full information on some one situation, 3) complete knowledge of all relevant laws, 4) correct insights into the basic situation, 5) sufficient skill in the manipulation of mathematical expressions, 6) correct insights into deduced situations, and 7) no restriction on the amount of time allowed for the deduction Then from the given situation the occurrence and the dimensions of the next significantly different situation can be deduced. Correct insights into the deduced data on this situation make it possible to deduce the occurrence and the dimensions of the third significantly different situation. Finally, since this procedure can be repeated indefinitely and since there are no restrictions on the amount of time to be devoted to the deduction, it makes no difference how many significantly different ~~sit~~ situations there are.

Fourthly, in a number of manners non-systematic process exhibits coincidental aggregates. For an aggregate is coincidental if 1) the members of the aggregate have some unity based on spatial juxtaposition or temporal succession or both and 2) there is no corresponding unity on the level of insight and intelligible relation.

For non-systematic process as a whole possesses a spatio-temporal unity but has no corresponding unity on the level of insight or intelligible relation.

Again, the several insights by which the several parts of non-systematic process are understood form another coincidental aggregate. For they are a multiplicity on the level of intelligibility but they possess some unity from the spatio-temporal unity of the process.

Similarly, the succession of different premises by which different stages of non-systematic process may be deduced are a third coincidental manifold. For they too are a multiplicity on the level of intelligibility but they possess some unity from the spatio-temporal unity of the process.

Further, the basic situation of non-systematic process must be a coincidental manifold. For it has unity by spatial juxtaposition; but it cannot be one on the level of insight and intelligible relation. If the basic situation were intelligibly one, then the deduction of the process from that intelligible unity would constitute an orderly grouping for the set of different insights and for the succession of different combinations of selected laws. But both the set of different insights and the succession of different combinations of selected laws are coincidental aggregates that cannot be unified by any orderly series or progression or grouping whatever. Therefore, the basic situation can be no more than a merely ~~[illegible]~~ spatial unification of different intelligibilities that can be grasped only by a set of different and unrelated insights.

Similarly, if many different and unrelated insights are needed to understand the basic situation, the premises for a deduction from that situation cannot be a single, unified combination of selected laws. And since a coincidental aggregate of premises will yield a coincidental aggregate of conclusions, it follows that every deducible situation, provided it is a total situation, also will be a coincidental aggregate. Further, it follows that, when a non-systematic process happens to give rise to a systematic process (as in recent theories on the origin of planetary systems), then the total situation must divide into two parts of which one happens to fulfil the ~~[illegible]~~ conditions of systematic process and the other fulfils the requirement of other things being equal.

Finally, there emerges the rule for constructing non-systematic processes. For "random" may be defined as "any whatever provided specified conditions of intelligibility are not fulfilled." But non-systematic process results from any basic situation provided it lacks intelligible unity from a definitive viewpoint. Therefore, the rule for constructing non-systematic processes is to begin from any random basic situation.

Fifthly, if non-systematic processes exist, then the difficulty of investigating their nature increases with the number and diversity of their several distinct and unrelated intelligibilities. Data on one situation are not equivalent to data on the whole process but are relevant only to one of many parts of the whole. Again, the types of data significant in one part will not be significant in disparate parts, and so several different inquiries must be undertaken. Thirdly, reports on one situation

[illegible] than a systematic

ordinarily cannot be checked by comparing them with inferences from reports on other situations. Fourthly, there is no supreme moment when all data fall into a single perspective, for there is ~~rmk~~ no single perspective to be had. Fifthly, even when the laws involved in the process are thoroughly understood, even when current and accurate reports from usually significant centers of information are available, still such slight differences in matters of fact can result in such large differences in the subsequent course of events that deductions have to be restricted to the short run and predictions have to be content with indicating probabilities. So, perhaps, it is that astronomers can publish the exact times of the eclipses of past and future centuries but meteorologists need a constant supply of fresh and accurate information to tell us about tomorrow's weather.

Let us now pause to take our bearings. We began by noting that concrete inferences from classical laws suppose not only knowledge of laws and information on some basic ~~in~~ situation but also an insight that mediates between the situation and general knowledge. We went on to distinguish between practical insights that apply laws to given situations and constructive insights that invent typical or ideal processes. We have been engaged in explaining that, just as constructive insight can devise systematic processes with all their beautiful and convenient properties, so also it can devise non-systematic processes with a complete set of quite opposite properties. It remains that a few more general corollaries be added.

First, systematic process is monotonous, but non-systematic process can be the womb of ~~gwarn~~ novelty. For the possibility of leaping deductively from any situation of a systematic process to any other situation rests on the fact that a systematic process is little more than a perpetual repetition of essentially the same story. On the other hand, the unfolding of a non-systematic process has to be followed through its sequence of situations. Significant changes ~~ma~~ occur and, as they occur, the relevant insights change. Hence, as will appear in Chapter IV, within a large non-systematic process there can be built a pyramid of schemes resting on schemes in a splendid ascent of novelty and creativeness.

Secondly, systematic process would seem to be reversible, that is, it would work equally well if, so to speak, the future were the past and the process ran backwards. For a systematic process is the expression of a single idea. Each successive situation is related to the next in accord with the dictates of the idea. Hence, to reverse the succession of dictates so that the process begins from a last situation and moves backwards to a first involves no new idea but merely a different and, it seems, equally workable application of the same idea. On the other hand, non-systematic process may easily be irreversible. For it is not the unfolding of some single idea, and successive situations are not related in accord with the dictates of any single insight or any single set of unified insights. What is in control is not intelligence but any random basic situation, and the resulting coincidental sequence of coincidental situations easily includes both the emergence and the destruction of systematic processes. Hence, to expect non-systematic process to be reversible is to expect destroyed systematic processes to re-emerge from their ruins; again, it is to expect that reversed systematic processes will resolve into their origins at the right moment and in the right manner though no provision is made for that resolution.

Thirdly, the distinction between systematic and non-systematic processes throws light on the precise meaning of closure. For there is an external closure that excludes outside interference. When it is applied to a systematic process, the whole course of events is mastered by intelligence with relative ease. But when
a it is applied to a non-systematic process, then it merely leaves internal factors all the freer to interfere with one another.

Fourthly, whether world process is systematic or non-systematic is a question to be settled by the empirical method of stating both hypotheses, working out as fully as one can the totality of their implications, and confronting the implications with the observable facts.

Fifthly, if world process proves to be non-systematic, then it contains coincidental aggregates and the word, "random," has an objective meaning. In that case, there would be some interpretation of statistical science as the science of what exists. In other words, in that case it would be false to say that statistical science must be a mere cloak for ignorance. Moreover, even if world process proves to be systematic, still that will be true only on empirical grounds and *a posteriori*; it follows that it cannot be true *a priori* that statistical science cannot be the science of what exists; On the present showing, then, ~~in other words~~, there can be no ~~xx~~ valid theoretical arguments that establish that statistical science in every possible meaning of the term must be a mere cloak for ignorance.

4. Statistical Heuristic Structure.

4.1 Elementary Contrasts.

Classical and statistical investigations exhibit marked differences that provide a convenient starting-point for the present section.

In the first place, while classical investigation heads towards the determination of functions and their systematization, statistical investigation clings to concrete situations. Hence, while classical conclusions are concerned with what would be if ~~sum~~ other things were equal, statistical conclusions directly regard such aggregates of events as the sequences of occasions on which a coin is tossed or dice are cast, the sequences of situations created by the mobility of molecules in a gas, the sequences of generations in which babies are born, the young marry, and the ~~ol~~ old die.

Secondly, statistical inquiry attends not to theoretical processes but to palpable results. As Galileo sought the intelligibility immanent in a free fall, so Clerk-Maxwell sought the intelligibility immanent in the electromagnetic field. But in a statistical investigation such theoretical analyses and constructions are set aside. The movement of dice observes perfectly the laws of mechanics, but the laws of mechanics are not premises in the determination of the probability of casting a "seven." Doctors commonly succeed in diagnosing the causes of death, but successful diagnoses are not studied in fixing death rates. The statistical scientist seems content to define events and areas, to count the instances of each defined class within the defined area, and to offer some general but rather vague view of things as a whole.

Thirdly, statistical science is empirical, but it does not endeavor to measure and correlate the spatial, temporal, and other variables that so fascinate classical investigators. Its attention is directed to frequencies that are straightforward numerical answers to the straightforward question, How often? Such frequencies may be ideal or actual but, while it is true that the ideal frequency or probability raises debatable issues, at least the actual frequency is a transparent report not of what should or might or will happen but of what in fact did happen. Such actual frequencies are absolute when they assign the actual number of events of a given kind within a given area during a given interval of time. However, since different areas commonly are not comparable, it is customary to proceed from absolute actual frequencies either to rates, say, per thousand of population or, when classes of events are alternative possibilities, to relative actual frequencies which are sets of proper fractions, say, p/n, q/n, r/n ..... where $n = p + q + r + ...$

Fourthly, behind the foregoing rather superficial differences, there is a profound difference in the mentality of classical and statistical inquirers. Had astronomers been content to regard the wandering of the planets as a merely random affair, the planetary system never would have been discovered. Had Joule been content to disregard small differences, the mechanical equivalent of heat would have remained unknown. But statistical inquirers make it their business to distinguish in their tables of frequencies between significant and merely random differences. Hence, while they go to great pains to arrive at exact numbers, they do not seem to attempt the obvious next step of exact explanation. As long as differences in frequency oscillate about some average, they are esteemed of no account; only when the average itself changes, is intellectual curiosity aroused and further inquiry deemed relevant.

4.2 The Inverse Insight.

The existence of this radical difference in mentality demands an explanation, and the obvious explanation is the occurrence of something like an inverse insight. For an inverse insight has three characteristics: it supposes a positive object of inquiry; it denies intelligibility to the object; and the ~~dent~~ denial runs counter to spontaneous anticipations of intelligence. But the differences named random are matters of fact: they occur in frequencies determined by counting the events in a given class in a given area during a given interval of time. Further, random differences are denied intelligibility for, though statistical inquirers hardly would use such an expression, at least their deeds seem a sufficient witness to their thought. When differences are not random, further inquiry is in order; but when differences are random, not only is no inquiry attempted but also the very attempt would be pronounced silly. Finally, this denial of intelligibility is in open conflict with the anticipations of classical investigation. For classical precept and example tirelessly inculcate the lesson that no difference is to be simply neglected; and while one may doubt that this classical attitude is more spontaneous than its opposite, at least one can speak of a devaluated inverse insight that divides classical and statistical anticipations.

Further, while this devaluated inverse insight bears on the frequencies of events, it does not follow necessarily that the defect of intelligibility resides in single ~~s~~ events. Indeed, it seems quite possible to acknowledge random differences in frequencies and at the same time to maintain that single events are determinate, that they are not random, even that they are deducible. At least, the events must be determinate enough to be counted for, if they are not counted, there are no frequencies and so no random differences in frequencies. Again, one can acknowledge random differences in death rates without suggesting that single deaths were random or that doctors were unable to perform successful diagnoses. Finally, if single events need not be random, they may be deducible. For if it is possible to argue from effect to cause, from consequent to antecedent, it should be equally possible to move from cause to effect, from determining antecedent to determined consequent.

It seems, then, that if we are to discover a __fully general__ account of the meaning of random differences, we must look not to single events but to events as members of a group. So the question becomes, How can there be a defect in intelligibility in a group of events if each event singly is quite determinate, if none are random, and if one by one all may be deduced?

Fortunately, if not accidentally, our previous discussion of concrete inferences from classical laws offers a ready answer to this question. For knowledge of laws can be applied 1) to single events, 2) to systematic processes, and 3) to non-systematic processes. Moreover, just as the assertion of random differences in frequencies need not imply that single events are indeterminate or random or that they are not deducible, so also in a non-systematic process each event may be determinate, none need be random and sometimes at least, if time were not money, all could be deduced. Again, just as the assertion of random differences springs from a devaluated inverse insight, so too does the notion of a non-systematic process. For a non-systematic process is as positive an object of inquiry as any process; it is non-systematic inasmuch as it lacks the intelligibility that characterizes systematic process; and its properties are very surprising indeed when they are compared with what commonly Laplace is supposed to have meant when he claimed that any situation in world history could be deduced from any other.

The similarity of these two devaluated inverse insights provides an obvious clue and, to follow it up, let us consider the four statements: 1) statistical inquiry is concerned with coincidental aggregates of events; 2) statistical inquiry investigates what classical inquiry neglects; 3) statistical inquiry finds an intelligibility in what classical inquiry neglects; and 4) this intelligibility is denied when random differences are affirmed.

First, statistical inquiry is concerned with coincidental aggregates of events. For it is not concerned with the intelligibly grouped events of systematic process: there are no statistics on the phases of the moon or on the transit of Venus, and there are no random differences in ordinary astronomical tables. Again, it is not concerned with events taken singly. For each single event amounts to just one more or less in tables of frequencies and, in general, a difference of one more or one less may be regarded as random. Further, it is possible to discern random differences in some groups of events in which each event is determinate and deducible and no event is random. It remains, then, that the object of statistical inquiry is the coincidental aggregate of events, ~~this~~ that is, the aggregate of events that has some unity by spatial juxtaposition or by temporal succession or by both but lacks unity on the level of insight and of intelligible relation. In other words, statistical inquiry is concerned with non-systematic process.

Secondly, statistical inquiry investigates what classical inquiry neglects. For even if one grants that classical inquiry leads to the laws that explain every event, it remains that classical science rarely bothers to explain the single events of non-systematic process and, still less, does it offer any technique for the orderly study of groups of such events. Moreover, there are excellent reasons for this neglect. The deduction of each of the events of a non-systematic process begins by demanding more abundant and more exact information than there is to be had. It proceeds through a sequence of stages determined by the coincidences of a random situation. It has to postulate unlimited time to be able to assert the possibility of completing the deduction. It would end up with a result that lacks generality for, while the result would hold for an exactly similar non-systematic process, it commonly would not provide a safe basis for an approximation to the course of another non-systematic process with a slightly different basic situation. Finally, it would be preposterous to attempt to deduce the course of events for every non-systematic process. Not only would the foregoing difficulties have to be surmounted an enormous number of times but this Herculean labor would seem to be to no purpose. How could non-systematic processes be classified? How could one list in an orderly fashion the totality of situations of all non-systematic processes? Yet without such a classification and such a list, how could one identify given situations with situations contained in the extremely long deductions of the extremely large set of non-systematic processes?

Thirdly, statistical inquiry finds an intelligibility in what classical inquiry neglects. So far we have been concerned to stress the defect of intelligibility in non-systematic process. But a mere defect in intelligibility is not the basis of a scientific method. There is needed a complementary direct insight that turns the tables of the defect. Just as scientific generalization exploits the fact that individuality pertains to an empirical residue, just as the real numbers, the theory of continuous functions, and the infinitesimal calculus exploit the defect of intelligibility in the continuum, just as scientific collaboration is possible because particular places and particular times pertain to the empirical residue, just as the principle of inertia and the basic postulate of Special Relativity rest on an empirically residual aspect of constant velocity, so also statistical science is the positive advance of intelligence through the gap in intelligibility in coincidental aggregates of events.

Accordingly, besides the devaluated inverse insight that has been our concern hitherto, there is to be acknowledged in statistical science another basic moment that is positive and creative. Aristotle was quite aware of what we have named non-systematic process, for he contended that the whole course of terrestrial events was just a series of accidents. But to this devaluated inverse insight he failed to add the further creative moment. Instead of discovering statistical method, he attempted to account for the manifest continuity of the terrestrial series of accidents by invoking the continuous influence of the continuously rotating celestial spheres.

Fourthly, it is this further intelligibility that is denied when random differences are affirmed. For if the statistical investigator deals with non-systematic processes, he does not find the intelligibility of systematic process either in the differences he pronounces significant or in the differences he pronounces random. Again, to discover the intelligibility that statistical science finds in non-systematic process, we must look to the differences pronounced significant. It follows that differences in frequencies of events are random when they lack not only the intelligibility of systematic process but also the intelligibility of non-systematic process.

4.3 The Meaning of Probability.

Still the reader will be more interested in hearing what this intelligibility is than in being told that it is lacking in random differences. Its name, then, is probability but to grasp the meaning of the name is to reach an explanatory definition. Let us begin from the definition and then try to understand it.

Consider a set of classes of events, P, Q, R,... and suppose that in a sequence of intervals or occasions events in each class occur respectively $p_1, q_1, r_1, \ldots\ p_2, q_2, r_2, \ldots\ p_i, q_i, r_i, \ldots$ times. Then the sequence of relative actual frequencies of the events will be the series of sets of proper fractions, $p_i / n_i,\ q_i / n_i,\ r_i / n_i, \ldots$ where $i = 1, 2, 3, \ldots$ and in each case $n_i = p_i + q_i + r_i + \ldots$. Now if there exists a single set of constant proper fractions, say $p/n,\ q/n,\ r/n, \ldots$ such that the differences

$$p/n - p_i/n_i,\quad q/n - q_i/n_i,\quad r/n - r_i/n_i, \ldots$$

are always random, then the constant proper fractions will be the respective probabilities of the classes of events, the association of these probabilities with the classes of events defines a state, and the set of observed relative actual frequencies is a representative sample of the state.

The foregoing ~~pa~~ paragraph outlines a procedure in which the central moment is an insight. By that insight the inquirer abstracts from the randomness in frequencies to discover regularities that are ~~ext~~ expressed in constant proper fractions named probabilities. There results the solution of two outstanding methodological problems. Because the probabilities are to hold universally, there is solved the problem of reaching general knowledge of events in non-systematic processes. Because states are defined by the association of classes of events with corresponding probabilities, there is by-passed the problem of distinguishing and listing non-systematic processes. However, both the probabilities and the states they define are merely the fruits of insight. They are hypothetical entities whose existence has to be verified and, in fact, becomes verified in the measure that subsequent frequencies of events conform to probable expectations. In turn, this need of verification provides a simple formulation for the notion of a representative sample. For a set of relative actual frequencies is a representative sample if the probabilities to which they lead prove to be correct. On the other hand, a set of relative actual frequencies is not a representative sample if the probabilities to which they lead run counter to the facts. It follows that the selection of representative samples is the basic practical problem of statistical inquiry and, indeed, that its solution must depend not merely on a full theoretical development of statistical method but also on the general knowledge of individual investigators and on their insights into whatever specific issues they happen to be investigating.

Such, then, is the general context, but our concern must center on the insight by which intelligence leaps from frequencies to probabilities and, by the same stroke, abstracts from the randomness in frequencies. Now an insight is neither a definition nor a postulate nor an argument but a preconceptual event. Hence our aim must be to encourage in readers the conscious occurrence of the intellectual events that make it possible to know what happens when probability is grasped. First, then, we shall consider an easier insight that bears some general resemblance to insights into probability. Secondly, we shall consider an insight that occurs when a particular case of probability is understood. Thirdly, we shall move towards the general ~~a~~ heuristic structure within which the notion of probability is developed and methods of determining its precise content are perfected.

In the first place, the mathematical notion of limit bears a general resemblance to the notion of probability. Accordingly, let us consider the simple sum,

$$\underline{S} = 1/2 + 1/4 + 1/8 + \ldots. \qquad [\text{to } \underline{n} \text{ terms}]$$
$$= 1 - 1/2^n$$

where, as $\underline{n}$ increases, $\underline{S}$ differs from unity by an ever smaller fraction and so, by assigning $\underline{n}$ ever larger values, the difference between the sum, $\underline{S}$, and unity can be made as small as one pleases. In the limit then, when the number of terms in the series is infinite, the sum, $\underline{S}$, is unity. However, one cannot write out an infinite number of terms; one cannot even conceive each of an infinite number of terms. Moreover, while it is contradictory to suppose that an unending series is ended, still one can understand the principle on which each fraction in the series is constructed, one can tell whether or not any fraction belongs to the series, one can conceive as many of the fractions as one pleases, and one can grasp that the more terms there are to the series, the nearer the sum is to unity. Finally, there is no contradiction in thinking or speaking of <u>all</u> the terms in the series, and one can see that there is no point in bothering about explicit conception of the remainder because it contains nothing that is not already understood. Now advertence to this absence of further intelligibility in the remainder is the abstractive aspect of the insight that claims the whole series to be understood sufficiently in its content and in its properties for it to be summed and for the sum to be equated with unity.

But, like a mathematical limit, a probability is a number. Like a limit, a probability is a number that cannot be reached from the data of a problem without the intervention of an insight. Again, just as the limit we considered under consideration lies beyond more terms than can be conceived, so a probability lies concealed within the random oscillations of relative actual frequencies. Finally, just as intelligence can reach a limit by grasping that there is nothing further to be understood in the unconceived infinite remainder of further terms, so also intelligence can reach probabilities by abstracting from the random oscillations of relative actual frequencies to discover a set of universally valid constants.

In the second place, to move closer to our quarry, let us analyze the tossing of a coin ~~a~~ in the hope of generating the insight that pronounces the probability of "heads" to be one-half. The result, then, of a toss is either of the alternatives, "heads" or "tails." In any given instance the result might have been different if 1) the initial ~~pos~~ position of the coin had been different or 2) different linear and angular momenta had been imparted to it or 3) the motion had been arrested at a different point. Let us name these three the determinants of the result and direct our attention to the set of possible combinations of determinants.

First, the set is very large. For any of a very large group of initial positions can be combined with any of a very large group of initial linear and angular momenta; and any of ~~these~~ those combinations can be combined with any of a very large group of points of arrested movement.

Secondly, the set of possible combinations divides into two exactly equal parts. For whenever "heads" results, "tails" would have resulted if the coin had been turned over and exactly the same toss and catch had been executed. Similarly, whenever "tails" results, "heads" would have resulted if the coin had been turned over and exactly the same toss and catch had been executed.

Thirdly, every sequence of actual combinations is a random selection from the set of possible combinations. It is a selection inasmuch as it need not include all possible combinations. It is a random selection inasmuch as it ~~is~~ may be any whatever provided specified conditions of intelligibility are not fulfilled. Now intelligibility is to be excluded not from single tosses but from the sequence of tosses as a sequence. It is not to be excluded from single tosses for there is no reason to suppose that tossing a coin involves a suspension of the laws of mechanics or of any similar science. It is to be excluded from the sequence as a sequence for we have every reason to assert that ~~tossing a coin~~ a sequence of tosses is not a systematic process. Hence, every sequence of actual combinations of determinants is a coincidental aggregate. It will possess the unity of a temporal succession. But while any single combination may be deducible from prior events, any sequence of combinations is deducible ~~as~~ only from some prior coincidental aggregate; for the sequence cannot be orderly in the sense that there is some insight or some set of unified insights that can be expressed in general terms and can determine the exact content of the sequence.

Now the relative actual frequency of "heads" is the fraction obtained by dividing the number of times "heads" occurs on any given succession of tosses by the number of tosses in that succession. Clearly, this fraction can and often will differ from one-half. For the result of each toss is settled by the actual combination of determinants, and that combination may be any combination whatever. However, differences between relative actual frequencies and one-half must be a coincidental aggregate. For if they were not, they would form an orderly series; if the differences formed an orderly series, the results would have to form an orderly series; if the results formed an orderly series, the sequence of combinations of determinants would form an orderly series. Ex hypothesi, this conclusion is false; therefore, the supposition was false. Moreover, relative actual frequencies cannot help oscillating about one-half. For the set of possible combinations divides into two exactly equal parts; and every sequence of actual combinations is a random selection from the set of possible combinations. Now in a random selection of a sequence the sequence is stripped of all order, all regularity, all law; hence, while it can and will include runs of "heads" and runs of "tails," it cannot possibly stick to one alternative to the exclusion of the other, and so relative actual frequency is bound to oscillate about one-half.

It has been shown that the relative actual frequencies of "heads" 1) can and often do differ from one-half but 2) only at random and 3) in a manner that yields an oscillation about one-half as a center. Intelligence, then, can grasp a regularity in the frequencies by abstracting from their random features and by settling on the one center about which they oscillate. That abstractive grasp of intelligibility is the insight that is expressed by saying that the probability of "heads" is one half.

However, it is only in games of chance that there can be discerned an antecedent symmetry in the set of possible combinations of determinants, of events. In other instances probabilities have to be reached *a posteriori* and, to reach them, a statistical heuristic structure has to be developed. To this issue we turn in the next subsection not, indeed, in the hope of determining what precisely probability must be in all cases but rather with the intention of grasping the underlying anticipations that inform statistical inquiry and are to be expected gradually to mount through trial and error, through theoretical discoveries and developing techniques, to some rounded methodological position such as already is enjoyed in classical investigations. In other words, besides the methodical genesis of scientific insights, there is the genesis of scientific method itself and, when a satisfactory account of the former is still a matter of obscure debates, a study of human understanding can draw no less profit from a consideration of the latter.

4.4 Analogy in Heuristic Structure.

The present subsection is a protracted analogy. Under ten successive headings we shall recall distinctive features of classical heuristic structure, note their reason or ground, and in each case proceed to an analogous feature in a statistical heuristic structure.

First, then, there is the unspecified heuristic concept. For the goal of every inquiry is an act of understanding, and the basic device of methodical inquiry is to name the unknown that will become known when the anticipated act of understanding occurs. Hence, just as the classical inquirer seeks to know the "nature of ..." so the statistical inquirer will seek to know the "state of ...."

Secondly, there is a specification of the heuristic concept by prescientific description. For all empirical inquiry presupposes some object that already is given but as yet is not understood; and every such object possesses its prescientific description that provides an initial specification for the heuristic concept. Hence, just as classical inquiry comes to know natures by understanding "data of different kinds," so statistical inquiry comes to know states by understanding "ordinary and exceptional, normal and abnormal runs of events."

Thirdly, linking the open heuristic concept with the prescientifically described object there is the heuristic theorem. Because similars are understood similarly, natures are linked with data classified by sensible similarity. So we speak of the nature of color or the nature of sound. Similarly, because a notable regularity is compatible with random differences in runs of events, states are linked with runs that despite occasional lapses are ordinary or normal or, again, with runs that are pronounced exceptional or abnormal though they contain a few ordinary or normal elements. So we speak of the state of a person's health, brokers speak of the state of the market, and the President of the United States discourses on the state of the nation.

Fourthly, to effect a transformation of prescientific anticipations and descriptions, there has to be formulated an ideal of scientific explanation. Hence, just as the classical inquirer places knowledge of nature in the discovery and verification of determinate functional relations, so the statistical inquirer places knowledge of states in the association of sets of classes of events with corresponding sets of probabilities. In other words, just as the mysterious nature of gravity turns out to be for the scientist merely a constant acceleration, so the mysterious state of so-and-so's health turns out to be for the scientist a schedule of probabilities attached to a schedule of classes of events.

Fifthly, from the formulation of the precise scientific objective there follows the displacement of prescientific by scientific description. Thus, to determine functional relations measurement is added to observation and mere sensible similarity gives way to similarities of conjunction and separation, of proportion and concomitant variation. In like manner to determine sets of probabilities the adjectives, ordinary and exceptional, normal and abnormal, are replaced by actual counting of events and the consequent tabulation of rates or of relative actual frequencies. Moreover, to justify this numerical accuracy, exact classifications are borrowed from classical science and every resource is employed to delimit, as far as possible, internally homogeneous volume-intervals of events.

Sixthly, just as classical inquiry derives a general view of its possibilities from the mathematical investigation of functions and of spatio-temporal relations, so statistical inquiry finds similar guidance and orientation in the calculus of probabilities.

Seventhly, just as classical inquiry evolves practical techniques of curve-fitting to aid the transition from measurements to functional relations, so statistical inquiry develops similar techniques to aid the transition from relative actual frequencies to probabilities.

Eighthly, just as classical inquiry proceeds not only from below upwards from measurements through curve-fitting but also from above downwards from differential equations to their solutions, so also a comparable department of statistical inquiry has discovered that the solution of operator equations yields eigenfunctions and eigenvalues that serve both to select classes of events and to determine the respective probabilities of the selected classes.

Ninthly, just as classical discovery is a leap of constructive intelligence that goes beyond ascertained measurements to posit a functional relation on which the relations between all appropriate subsequent measurements should converge as on a limit, so also statistical discovery (as distinct from statistical information) is a leap of constructive intelligence that goes beyond ascertained relative actual frequencies to assign probabilities where differences between probabilities and relative actual frequencies 1) should always be a coincidental aggregate and 2) in each case should be eliminable by extending the investigation of that case.

Hence, just as classical laws are universal and constant while measurements are particular and subject to the variations introduced by extraneous influences, so statistical states are universal and constant though relative actual frequencies are particular and subject to random differences.

However, while both types of discovery are universal and so abstract, still they involve different types of abstraction. In both classical and statistical constructs there is abstraction from the empirically residual aspects of individuality, of the continuum, of particular places and times, and of constant velocity. But classical laws, at least in the determination of each law, also abstract from coincidental aggregates inasmuch as they demand the qualification, "other things being equal." On the other hand, statistical states express an intelligibility immanent in coincid- ~~coincid~~ ental aggregates and, to reach this intelligibility, they abstract from the random differences in relative actual frequencies.

Tenthly, no less than the classical law, the statistical state has to be verified. For knowledge of states is derived from particular frequencies by a leap of constructive intelligence. That leap is neither the recognition of a fact nor the grasp of a necessity but simply an insight into possibility. The known frequencies are ~~m~~ satisfied by the supposition of a state that universally is manifested by events of determinate classes occurring with determinate probabilities. But further investigation can ~~confirm~~ compromise this result in a variety of manners. It may reveal an unsatisfactory classification of events, an underestimation of the complexity of the sequence of situations, a failure to reach representative samples. Then relative actual frequencies have to be ascertained on a more exact or broader basis, and the constructive leap has to be repeated in a new manner.

Still though both classical and statistical hypotheses need verification, verification has not the same meaning in both cases. Because the relations between measurements converge on the functional relations that express classical laws, it is possible to substitute the numerical values determined by the measurements for the variables that are functionally related by the laws. In contrast, because relative actual frequencies differ at random from probabilities, it is not possible to deduce the probabilities from any fully determinate mathematical formula by substituting for the variables of the formula the fractions that correspond to relative actual frequencies.

The converse to this difference in the meaning of verification appears in the difference between classical and statistical predictions. Classical predictions can be exact within assignable limits, because relations between measurements converge on the functional relations that formulate classical laws. But because relative actual frequencies differ at random from probabilities, statistical predictions primarily regard the probabilities of events and only secondarily determine the corresponding frequencies that differ at random from the probabilities. Hence, even when numbers are very great and probabilities high, as in the kinetic theory of gases, the possibility of exceptions has to be acknowledged; and when predictions rest on a statistical axiomatic structure, as in quantum mechanics, the structure itself seems to involve a principle of indeterminacy or uncertainty.

4.5 Some Further Questions.

Possible further questions abound. But as the shrewd reader will have surmised, our purpose has been not to work out definitive foundations for statistical science but to grasp in some fashion the statistical heuristic structure that not only tackles specific problems but also develops its own methods as it goes along and thereby sets up an exigence for a succession of new and better foundations.

In particular there will be noticed a certain looseness in the notions of state and of probability. But it is not indeliberate. The intelligent formulation of any notion is the fruit of an insight, and insights grasp not only necessities but also mere possibilities. There is an insight that leads to the definition of the circle, but it does not prove that circles exist. There is a cluster of insights that ~~is~~ are formulated in Euclidean geometry, but they do not prove the existence of Euclidean space. Similarly, there is a rather complex insight that leads to a notion of probability, and there is a cluster of insights expressed in a calculus of probabilities. But the excellence of the insights and the intellectual satisfaction they yield do not establish their correspondence with the specific content of verifiable probabilities and verifiable relations between probabilities. At least, I do not see my way to excluding on the general level of this inquiry the possibility that a range of different fields of relations between probabilities may be formulated and that statistical science may have the task of selecting one of these fields of relations and the type of probability they define implicitly.

Again, I may be asked for the operational meaning of the highly theoretical coincidental aggregate. The answer is that the appropriate operation occurs on the methodological level. Either a range of ~~general~~ observations ~~are~~ is to be subsumed under classical heuristic structure or ~~they are~~ it is to be subsumed under statistical heuristic structure. On the former hypothesis it will be possible to discover some orderly series, progression, or grouping. On the latter hypothesis no such series, progression, or grouping exists. Both hypotheses can be formulated; their implications are to be worked out; and the facts are to decide which hypothesis is, if not ultimate truth, at least the best available opinion at the given stage of scientific development.

Finally, if probabilities must be verified, it also is true that there is a probability of verifications. But it is of no little importance that this second probability shares the name but not the nature of the first. For the first probability apart from random differences, corresponds to the relative actual frequency of events. It is the regularity in the frequencies and it is to be known by a leap of constructive intelligence that grasps the regularity by abstracting from the randomness. In contrast, the second probability is not some fraction that, apart from random differences, corresponds to the relative actual frequency of verifications. A preponderance of favorable tests do not make a conclusion almost certain; indeed, a very few contrary tests suffice to make it highly improbable. More fundamentally, the second probability is not known by a leap of constructive intelligence that abstracts from random differences, for such leaps never yield anything but hypotheses. As will appear in Chapters IX and X, the second probability is known through acts of reflective understanding and judgment; it means that an affirmation or negation leads towards the unconditioned; and it is estimated, not by counting verifications and abstracting from random differences, but by criticizing verifications and by taking everything relevant into account.

For these reasons, then, we distinguish sharply between "probably occurring" and "probably true." For the same reasons we refuse to identify "certainty" in the sense of unit probability with "certainty" in the sense of "certainly verified." It follows that we find it meaningless to represent by a fraction the probability of a verification. Similarly, we find it fallacious to argue that probable events are not certain events because probable judgments are not certain judgments. Indeed, that fallacy ~~would wreck our whole analysis: for we have granted that single events may be deducible and, in that sense, certain yet the same events as a group may form a coincidental aggregate and so, when investigated with the generality made possible by statistical~~

would wreck our analysis. Not only are there two meanings to probability and two meanings to certainty but also there are two manners in which some events of non-systematic process can be investigated. Classical procedures would yield particular, probably verified conclusions about single events assigned a unit probability, where statistical procedures would yield general, probably verified conclusions about events as members of coincidental aggregates by assigning them fractional probabilities.

Before closing it may be well to add a word on the use of the terms, "classical" and "statistical." In contemporary physics it is customary to oppose "classical" to "quantum" and "statistical" to "mechanical." So there arises the familiar division of classical mechanics (Newton), classical statistics (Boltzmann), quantum mechanics (Schrödinger, Heisenberg), and quantum statistics (Bose-Einstein, Fermi-Dirac). Clearly, however, the present study of heuristic structures demands not a fourfold but a twofold division. Either intelligence anticipates the discovery of functional relations on which relations between measurements will converge, or else it anticipates the discovery of probabilities from which relative actual frequencies may diverge though only at random. The latter alternative has a fairly clear claim to the name, "statistical." The former alternative is not limited to Newtonian mechanics and, in the opinion of many, does not regard quantum mechanics. It is a mode of inquiry common to Galileo, Newton, Clerk-Maxwell, and Einstein; it is as familiar to the chemist as to the physicist; it long was considered the unique mode of scientific investigation; it has been the principal source of the high repute of science. In such a work as the present no one, I trust, will be misled if so classical a procedure is named "classical."

5. Survey.

Perhaps enough progress has been made for the rather novel orientation of this inquiry to come into better focus. We began from the description of a discovery to proceed to distinguish insights, their cumulation to higher viewpoints, and the significance of grasping that at times the point is that there is no point. In the present chapter we have moved not forward and outward to conclusions about objects but rather backward and inward to the subject's anticipations of insights that have not occurred and to the methodical exploitation of such anticipations. In that inward movement the reader can foresee the direction in which the whole work will advance. For our goal is not any scientific object, any universal and necessary truth, any primary propositions. Our goal is the concrete, individual, existing subject that intelligently generates and critically evaluates and progressively revises every scientific object, every uncautious statement, every rigorously logical resting place that offers prematurely a home for the restless dynamism of human understanding. Our ambition is to reach neither the known nor the knowable but the knower. Chapter I spoke of the insights he seeks. Chapter II has introduced the heuristic structures that inform his seeking. Chapters III to V will consolidate this position. Chapters VI and VII will turn to the activities of more or less intelligent common sense. Chapter VIII will bring science and common sense together. Chapters IX and X will tackle the problems of critical judgment ~~to~~ and, incidentally, will explain to impatient readers what they have been about while we in the first eight chapters were attempting to communicate to them the necessary prior insights. Chapters XI to XVII endeavor to grasp within a single view the totality of views on knowledge, objectivity, and reality, for all proceed from the empirical, intellectual, and rational consciousness of the concrete subject.

I do not know whether this remains or not.

Appendix to Chapter II.

On the Use of the Terms "Classical" and "Statistical."

In ordinary usage, "classical" and "statistical" are not opposed. The opposite to "classical" is "quantum," and the opposite to "statistical" is "mechanical." This usage may be illustrated by the fourfold classification of 1) classical mechanics (Newton), 2) classical statistics (Boltzmann), 3) quantum mechanics (Schrödinger, Heisenberg), and 4)~~q~~ quantum statistics (Bose-Einstein, Fermi-Dirac).

The trouble is that this fourfold classification seems incomplete. For relativity mechanics is opposed to classical mechanics and, while special relativity enters into combination with quantum mechanics (Dirac), general relativity seems as opposed to it as Einstein himself. Further, if these complications are not to be neglected, it is necessary to go behind the terminology to a systematic conception of the conceptions entertained by interpreters of physical theory. As is obvious, however, the purpose of this appendix is not to expound and to justify a systematic ~~view but simply to clarify our linguistic usage~~ view but simply to clarify the linguistic usage that we have found convenient by contrasting its assumptions with the assumptions that seem to underlie more common modes of speech.

From our viewpoint, then, the fundamental disjunction regards the interpretation of laws of the Newtonian and Einsteinian type. Such laws will be said to be interpreted concretely if they are taken to relate imaginable terms. The same laws will be said to be interpreted abstractly if they are taken to relate terms that are defined implicitly by the laws themselves.

On the first alternative of concrete interpretation, the law is completely determinate in principle. It is true enough that the law is expressed by a mathematical formula of wide generality and that further determinations will have to be added before any application to concrete instances can occur. It also is true that the further determinations cannot be deduced from the law as a mathematical or as a physical formula. But on concrete interpretation the law is not simply a physical formula; it relates imaginable terms; and because terms are imaginable inasmuch as their various dimensions are assignable, it follows that for concrete interpretation the law is fully determinate in principle.

However, those that accept the first alternative split into two groups. The first group not only affirms concrete interpretation but also affirms that concretely interpreted laws of the Newtonian type exist. The second group agrees with the first in admitting concrete interpretation but differs from it by affirming that, if any such laws seem to be verified, the verification is mere macroscopic appearance. The agreement and difference of this first and this second group seem to me to correspond to the agreement ~~and the difference that that unites and the difference that separates classical statistics and quantum mechanics.~~ that unites and the difference that separates ordinary conceptions of classical ~~mechanics and~~ statistics and quantum mechanics.

On the second alternative of abstract interpretation, laws of the Newtonian and Einsteinian type are determinate but not fully determinate. They are determinate in their own abstract order. They are not fully determinate in two respects. In the first place, they are applied to concrete instances only by assigning precise numerical values to be substituted for their variables; and from abstract laws alone such precise numerical values cannot be deduced. In the second place, when one deduces some precise numerical values from other known numerical values, the deduction rests not simply on the truth of the abstract laws but also on the truth that such and such a conjunction of abstract laws is alone relevant to such and such a concrete process in such and such a concrete situation.

Now it is this second indeterminacy of abstract laws that is significant. For it is an indeterminacy that resides primarily not in the knowing subject but in the object ~~to be known. Thus, to affirm the existence of a planetary~~ system removes an indeterminacy in knowing, for it posits such and such a conjunction of laws as alone relevant to a given concrete process in a given concrete situation. Still, this removal of an indeterminacy in knowing rests, not on abstract laws alone, but on a set of concrete matters of fact; it rests on matters of fact that might be otherwise; and if the matters of fact were otherwise, the extraordinarily accurate predictions of astronomers would vanish and in their place ~~there would be merely the unsolved 3-body problem~~

there would arise an insoluble problem in which the selection of relevant laws would depend on a diverging series of observations and the selection of the rele~~vant observations would~~ depend on ~~knowledge of the relevant laws~~.

to be known. For situations are of two kinds. In some concrete situations the relevant laws are concretely applicable in a closed or almost closed circle of mutual conditioning; and such is the case in the planetary system that has provided the most striking examples of accurate deduction and prediction. But there are other concrete situations in which the relevant laws are applicable only in a diverging and scattering series of ever more numerous and more remote conditions; and such would be the case if one attempted to deduce and predict the emergence or destruction of planetary systems.

Hence, from the viewpoint of abstract interpretation, one distinguishes between 1) abstract laws applied to schematic situations and 2) abstract laws applied to non-schematic situations. In both cases the abstract laws exist and govern every event that occurs; in neither case are the abstract laws mere macroscopic appearance. But in the first case accurate ~~deduction and prediction are possible. In the second case there is no objective possibility of accurate deduction and prediction because there is no objective mediation between scheme that removes the indeterminacy of the abstract laws.~~ deduction and prediction are possible objectively because the situation is schematic. And in the second case there is no possibility of removing the native indeterminacy of the abstract laws and proceeding to accurate deductions and predictions because the objective condition of a schematic situation is not fulfilled.

The arguments for this abstract interpretation will appear in the course of Chapter III. It differs from the interpretative assumptions of classical mechanics and statistics inasmuch as 1) it restricts predictable events to schematic situations and 2) it denies schematic situations to be the sole situati ns that exist. It differs from the interpretative assumptions associated with quantum mechanics and statistics inasmuch as it rejects as invalid the inf rence that, because there are non-schematic situations in which predictions are not objectively possible, therefore laws of the Newtonian and Einsteinian type are mere macroscopic appearance. Finally, ~~it is~~ in the light of this abstract ~~interpretation that we refer to laws of the Newtonian and Einsteinian type as classical~~ interpretation, we feel justified in continuing to refer to laws of the Newtonian and Einsteinian type as classical and in opposing to them solely the statistical laws that have to be invoked in dealing with non-schematic situations.