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Insight: Elements.

In the midst of that vast and profound stirring of human minds, which we name the Renaissance, Descartes was convinced that too many people felt it beneath them to direct their efforts to apparently trifling problems. Again and again, in his Régulae ad directionem ingenii, he reverts to this theme. Intellectual mastery of mathematics, of the departments of science, of philosophy, is the ~~slowly accumulated~~ fruit of a slow and steady accumulation of little insights. Great problems are solved by being broken down into little problems. The strokes of genius are but the outcome of a continuous habit of inquiry that grasps clearly and distinctly all that is involved in ~~ever~~ the simple things that anyone can understand.

I thought it well to begin by recalling this conviction of a famous mathematician and philosopher for our first task will be to attain familiarity with ~~a~~ what is meant by insight and the only way to achieve this end is, it seems, to attend very closely to a series of instances all of which are rather remarkable for their banality.

1. The psychological resonance of the occurrence of an insight appears, with the over-emphasis that aids clarity, in the story of Archimedes running from the bath with the cry, Eureka. He had been set the problem of determining whether a crown, fashioned by a smith of doubtful honesty, contained only gold. He had grasped, suddenly and unexpectedly, what we would name the principles of ~~human~~ buoyancy and of specific gravity. Still he did not grasp these principles in some abstract formulation Rather, he had understood that he would solve his problem by weighing the crown in water.

In the story of Archimedes' discovery, the key-note is delight and joy. I do not suppose that the history of science contains another instance of such dramatic

Though there have been ~~other and greater discoveries, certainly Archimedes~~

1. With the over-emphasis that aids clarity, ~~I~~ the psychological resonance of the occurrence of an insight appears in the story of Archimedes rushing naked from the baths of Syracuse with the cryptic cry, Eureka. A smith of doubtful votive/ honesty had fashioned a crown for King Hiero. The king set Archimedes the problem of determining whether it had been made of pure gold. Archimedes had hit upon a solution, Weigh the crown in water.

If there have been greater discoveries in the history of ~~science, there is not a more~~ uninhibited expression of the delight and joy that the solution of a problem brings.

A Dramatic Instance

1\. Our first illustrative instance of insight will be the story of Archimedes rushing naked from the baths of Syracuse with the cryptic cry, Eureka! King Hiero, it seems, had had a votive crown fashioned by a smith of rare skill and doubtful honesty. He wished to know whether or not baser metals had been added to the gold. Archimedes was set the problem and in the bath had hit upon the solution. Weigh the crown in water! Implicit in this directive were the principles of displacement and of specific gravity.

With those principles of hydrostatics we are not directly concerned. For our objective is an insight into insight. Archimedes had his insight by thinking about the crown; we shall have ours by thinking about Archimedes. What we have to grasp is that insight 1) comes as a release to the tension of inquiry, 2) comes suddenly and unexpectedly, 3) is a function not of outer circumstance but inner conditions, 4) pivots between the concrete and the abstract, and 5) passes into the habitual texture of one's mind.

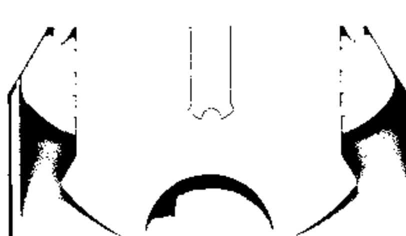
First, then, insight comes as a release to the tension of inquiry. This feature is dramatized in the story by Archimedes' peculiarly uninhibited exultation. ~~But it is, I think, merely to miss the point if one argues that Archimedes' behavior is hardly typical of scientists~~ ~~No doubt, such an outburst of delight is not typical, scientific behavior~~ But the point I would make does not lie in this outburst of delight but in the antecedent desire and effort that it betrays. For if the typical scientist's satisfaction in success is more sedate, his earnestness in inquiry can still exceed that of Archimedes. Deep within us all, emergent when the noise of other appetites is stilled, there is a drive to know, to understand, to see why, to discover the reason, to find the cause, to explain. Just what is wanted, has many names. In what precisely it consists, is a matter of dispute. But the fact of inquiry is beyond all doubt. It can absorb a man. It can keep him for hours, day after day, year after year, in the narrow prison of his study or his laboratory. It can send him on dangerous voyages of exploration. It can withdraw him from other interests, other pursuits, other pleasures, other achievements. It can fill his waking thoughts, hide from him the world of ordinary affairs, invade the very fabric of his dreams. It can demand endless sacrifices that are made without regret though there is only the hope, never a certain promise, of success. What better symbol could one find for this obscure, exigent, imperious drive than a man, naked, running, excitedly crying "I've got it."

Secondly, insight comes suddenly and unexpectedly. It did not occur when Archimedes was in the mood and posture that a sculptor would select to portray "The Thinker." It came in a flash, on a trivial occasion, in a moment of relaxation. Once more there is dramatized a universal aspect of insight. For it is reached, in the last analysis, not by learning rules, not by following precepts, not by studying any methodology. Discovery is a new beginning. It is the origin of new rules that supplement or even supplant the old. Genius is creative.

It is genius precisely because it disregards established routines, because it originates the novelties that will be the routines of the future. Were there rules for discovery, then discoveries would be mere conclusions. Were there precepts for genius, then men of genius would be hacks. Indeed, what is true of discovery, also holds for the transmission of discoveries by teaching. For a teacher cannot undertake to make a pupil understand. All he can do is present the sensible elements in the issue in a suggestive order and with a proper distribution of emphasis. It is up to the pupils themselves to reach understanding, and they do so with varying measures of ease and rapidity. Some get the point before the teacher can finish his exposition. Others just manage to keep pace with him. Others see the light only when they go over the matter by themselves. Some finally never catch on at all; for a while they follow the classes but, sooner or later, they drop by the way.

Thirdly, insight is a function not of outer circumstances but of inner conditions. Many frequented the baths of Syracuse without coming to grasp the principles of hydrostatics. But who bathed there without feeling the water or without finding it hot or cold or tepid? There is, then, a strange difference between insight and sensation. Unless one is deaf, one cannot avoid hearing. Unless one is blind, one has only to open one's eyes to see. The occurrence and the content of sensation stand in some immediate correlation with outer circumstance. But with insight internal conditions are paramount. Thus, insight depends upon native endowment&so, with fair accuracy, one can say that insight is the act that occurs frequently in the intelligent and rarely in the stupid. Again, insight depends upon a habitual orientation, upon a perpetual alertness ever asking the little question, Why? Finally, insight depends on the accurate presentation of definite problems. Had Hiero not put his problem to Archimedes, had Archimedes not thought earnestly, perhaps desperately, upon it, the baths of Syracuse would have been no more famous than any others.

Fourthly, insight pivots between the concrete and the abstract. Archimedes' problem was concrete. He had to settle whether a particular crown was made of pure gold. Archimedes solution was concrete. It was to weigh the crown in water. Yet if we ask what was the point to that procedure, we have to have recourse to the abstract formulations of the principles of displacement and of specific gravity. Without that point, weighing the crown in water would be mere eccentricity. Once the point is grasped, King Hiero and his golden crown become minor historical details of no scientific importance. Once more the story dramatizes a universal aspect of insight. For if insights arise from concrete problems, if they reveal their value in concrete applications, none the less they possess a significance greater than their origins and a relevance wider than their original applications. Because ~~they~~ insights arise with reference to the concrete, geometers use diagrams, mathematicians invent ~~symbols, teachers need blackboards, doctors have to see their patients, trouble-shooters have to~~
symbols, teachers need black-boards, pupils have to perform experiments for themselves, doctors have to see their patients,



It is genius precisely because it disregards ~~every~~ established routine, because it originates the routines of the fut

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trouble-shooters have to travel to the spot, people with a mechanical bent take things apart to see how they work. But because the significance and relevance of insight goes beyond any concrete problem or application, men formulate abstract sciences with their numbers and symbols, their technical terms and formulae, their definitions, postulates, and deductions. Thus, by its very nature, insight is the mediator, the hinge, the pivot. It is insight <u>into</u> the concrete world of sense and imagination. Yet what is known by insight, what insight adds to sensible and imagined presentations, finds its adequate expression only in the abstract and recondite formulations of the sciences.

Fifthly, insight passes into the habitual texture of one's mind. Before Archimedes ~~had~~ could solve his problem, he needed an instant of inspiration. But he needed no further inspiration when he went to offer the king his solution. Once one has understood, one has crossed a divide. What a moment ago was an insoluble problem, now becomes incredibly simple and obvious. Moreover, it tends to remain simple and obvious. However laborious the first occurrence of an insight may be, subsequent repetitions occur almost at will. This, too, is a universal characteristic of insight and, indeed, it constitutes ~~the possibility of learning. For in learning a subject there is an initial period of darkness in which one gropes about insecurely and then, as one begins to catch on, there is a subsequent period of increasing light, confidence, interest, absorption. Moreover, this rule holds just in the degree that a subject calls, not for mere memory work, but for a grasp of principles, for the formation of new concepts,~~ the possibility of learning. For we can learn inasmuch as we can add insight to insight, inasmuch as the new does not extrude the old but complements and combines with it. Inversely, inasmuch as the subject ~~a subject~~ to be learnt involves the acquisition of a whole series of insights, the process of learning is marked by an initial period of darkness in which one gropes about insecurely, in which one cannot see where one is going, in which one cannot grasp what all the fuss is about; and only gradually, as one begins to catch on, does the initial darkness yield to a subsequent period ~~in~~ of increasing light, confidence, interest, absorption. Then, the infinitesimal calculus or theoretical physics or the issues of philosophy cease to be the mysterious ~~and foggy realms they had seemed; things become for us simple and obvious we too begin to find them less incredible our teachers' i claims that really such matters are, not at all impossible, but simple and obvious as simple and obvious as anything else is, once one has understood.~~ and foggy realms they had seemed. Imperceptibly we shift from the helpless infancy of the beginner to the modest self-confidence of the advanced student. Eventually we become capable of taking over the teacher's role and complaining of the remarkable obtusenes of pupils that fail to see what, of course, is perfectly simple and obvious, to those that understand.

2. Definition. As every school-boy knows, a circle is a locus of coplanar points equidistant from a center. What every school-boy does not know is the difference between repeating that definition, as a parrot might, and uttering it intelligently. So, with a sidelong bow to Descartes' insistence on the importance of understanding very simple things, let us inquire into the genesis of the definition of the circle.

~~Imagine a cart-wheel with its bulky hub, its stout spokes, its solid rim.~~

~~Ask a question. Why is it round? More precisely, rule out of consideration such extrinsic grounds of the wheel's roundness~~

~~Limit the question. One might explain the roundness of the wheel by appealing to its maker; the wheelwright because the wheelwright proceeded in such and such a fashion, his product had to be of such a kind. Again, one might seek explanation in appealing to the wheelwright's tools or to~~

2.1 The Clue. Imagine a cart-wheel with its bulky hub, its stout spokes, its solid rim.

Ask a question. Why is it round?

Limit the question. What is wanted is the immanent ground of the roundness of the wheel. Hence a correct answer will not introduce new data such as carts, carting, transportation, or wheelwrights, or their tools. It will appeal simply to the wheel.

Consider a suggestion. The wheel is round because its spokes are equal. Clearly, that will not do. The spokes could be equal yet sink unequally into the hub and rim. Again, the rim could be flat between successive spokes.

Still, we have a clue. Let the hub decrease to a point; let the rim and spokes thin out into lines; then, if there were an infinity of spokes and all were exactly equal, the rim would have to be perfectly round; inversely, were any of the spokes unequal, the rim could not avoid bumps or dents. Hence, we can say that the wheel necessarily is round, inasmuch as the distance from the center of the hub to the outside of the rim is always the same.

~~However, if this brings us close enough to the definition of the circle, it is only a preliminary to our proper objective. What we desire is an insight, not into the circle, but into the act illustrated by insight into the circle. Accordingly, a number of observations on~~

A number of observations are now in order. The foregoing brings us close enough to the definition of the circle. ~~But our purpose is not to attain insight into the circle~~ But our purpose is to attain insight, not into the circle, but into the act illustrated by insight into the circle.

The first observation, then, is that points ~~lines~~ and lines cannot be imagined. One can imagine an extremely small dot. But no matter how small a dot may be, still it has magnitude. To reach a point, all magnitude must vanish, and with all magnitude there vanishes the dot as well. One can imagine an extremely fine thread. But no matter how fine a thread may be, still it has breadth and depth as well as length. Remove from the image all breadth and depth, and there vanishes all length as well.

2.2 The Concepts. The second observation is that points and lines are concepts.

Just as imagination is the playground of our desires and our fears, so ~~imagination~~ conception is the playground of our intelligence. Just as imagination can create objects never seen nor heard nor felt, so too conception can create objects that cannot even be imagined. How? By supposing. The imagined dot has magnitude as well as position, but the geometer says, Let us suppose it has only position. The imagined line has breadth as well as length, but the geometer says, Let us suppose it has only length.

Still, there is method in this madness. Our images and especially our dreams seem very random affairs, yet psychologists offer to explain them. Similarly, the suppositions underlying concepts may appear very fanciful, yet they too can be explained. Why did we require the hub to decrease to a point and the spokes and rim to mere lines? Because we had a clue -- the equality of the spokes -- and we were pushing it for all it was worth. As long as the hub had any magnitude, the spokes could sink into it unequally. As long as the spokes had any thickness, the wheel could be flat at their ends. So we supposed a point without magnitude and lines without thickness to obtain a curve that would be perfectly, necessarily round.

Note, then, two properties of concepts. In the first place they are constituted by the mere activity of supposing, thinking, considering, formulating, defining. They may or may not be more than that. But if they are more, then they are not merely concepts. And if they are no more than supposed or considered or thought about, still that is enough to constitute them as concepts. In the second place, concepts do not occur at random; they emerge in thinking, supposing, considering, defining, formulating; and that many-named activity occurs, not at random, but in conjunction with an act of insight.

2.3 The Image. The third observation is that the image is necessary for the insight.

Points and lines cannot be imagined. But neither can necessity or impossibility be imagined. Yet in approaching the definition of the circle there occurred some apprehension of necessity and of impossibility. As we remarked, if all the radii are equal, the curve must be perfectly round; and if any radii are unequal, the curve cannot avoid bumps or dents.

Further, the necessity in question was not necessity in general but a necessity of roundness resulting from these equal radii. Similarly, the impossibility in question was not impossibility in the abstract but an impossibility of ~~roundness~~ roundness resulting from these unequal radii. Eliminate the image of the center, the radii, the curve, and by the same stroke there vanishes all grasp of necessary or of impossible roundness.

But it is that grasp that constitutes the insight. It is the occurrence of that grasp that makes the difference between repeating the definition of a circle, as a parrot might, and uttering it intelligently, uttering it with the ability to make up a new definition for oneself.

It follows that the image is necessary for the insight. Inversely, it follows that the insight is the act of catching on to a connection between imagined equal radii and, on the other hand, a curve that is bound to look perfectly round.

2.4 The Question. The fourth observation adverts to the question.
There is the question as expressed in words. Why is the wheel round?

Behind the words there may be conceptual acts of meaning, such as "wheel", "round," etc.

Behind these concepts there may be insights in which one grasps how to use such words as "wheel," "round," etc.

But what we are trying to get at, is something different. Where does the "Why?" come from? What does it reveal or represent? Already we had occasion to speak of the psychological tension that had its release in the joy of discovery. It is that tension, that drive, that desire to understand, that constitutes the primordial "Why?" Name it what you please, alertness of mind, intellectual curiosity, the spirit of inquiry, active intelligence, the drive to know. Under any name it remains the same and is, I trust, very familiar to you. ~~But~~

This primordial drive, then, is the pure question. It is prior to any insights, any concepts, any words, for insights, concepts, words have to do with answers; and before we look for answers, we want them; such wanting is the pure question.

On the other hand, though the pure question is prior to insights, concepts, and words, it presupposes experiences and images. Just as insight is into the concretely given or imagined, so the pure question is about the concretely given or imagined. It is the wonder, which Aristotle claimed to be the beginning of all science and philosophy. But no one just wonders. We wonder about something.

2.5 Genesis. A fifth observation distinguishes moments in the genesis of a definition.

When an animal has nothing to do, it goes to sleep. When a man has nothing to do, he may ask questions. The first moment is ~~the~~ an awakening to one's intelligence. It is release from the dominance of biological drive and from the routines of everyday living. It is the effective emergence of wonder, of the desire to understand.

The second moment is the hint, the suggestion, the clue. Insight has begun. We have got hold of something. There is a change that we are on the right track. Let's see.

The third moment is the process. Imagination has been released from other cares. It is free to cooperate with intellectual effort, and its cooperation ~~runs parallel~~ consists in endeavoring to run parallel to intelligent suppositions while, at the same time, restraining supposition within some limits of approximation to the imaginable field.

The fourth moment is achievement. By their cooperation, by successive adjustments, question and insight, image and concepts, present a solid front. The answer is a patterned set of concepts. The image strains to approximate to the concepts. The concepts, by added conceptual determinations, can express their difference from the merely approximate image. The pivot between images and concepts is the insight. And setting the standard which insight, images, and concepts must meet is the question, the desire to know, that could have kept the process in motion by further queries, had its requirements not been satisfie

2.6 Nominal and Explanatory Definition.

A sixth observation distinguishes different kinds of definition. As Euclid defined a straight line as a line lying evenly between its extremes, so he might have defined a circle as a perfectly round plane curve. As the former definition so also the latter would serve to determine unequivocally the proper use of the names, straight line, circle. But, in fact, Euclid's definition of the circle ~~ix~~ does more than reveal the proper use of the name, circle. It includes ~~what otherwise would have had to be added as a postulate,~~ the affirmation that in any circle all radii are exactly equal; and were that affirmation not included in the definition, then it would have had to be added as a postulate.

To view the same matter from another angle, Euclid did postulate that all right angles be equal. Let us name the sum of two adjacent right angles a straight angle. Then, if all right angles are equal, necessarily all straight angles will be equal. Inversely, if all straight angles are equal, all right angles must be equal. Now if straight lines are really straight, if they never bend in any direction, must not all ~~right~~ straight angles be equal? Could not the postulate of the equality of straight angles be included in the definition of the straight line, as the postulate of the equality of radii is included in the definition of the circle?

At any rate, there is a difference between nominal and explanatory definitions. Nominal definitions merely tell us about the correct usage of names. Explanatory definitions also include something further that, were it not included in the definition, would have to be added as a postulate.

What constitutes the difference? It is not that explanatory definitions suppose an insight while nominal definitions do not. For a language is an enormously complicated tool with an/almost/endless variety of parts that admit a far greater number of significant combinations. If insight is needed to see how other tools are to be used properly and effectively, insight is similarly needed to use a language properly and effectively.

Still, this yields, I think, the answer to our question. Both nominal and explanatory definitions suppose insights. But a nominal definition supposes no more than an insight into the proper use of language. An explanatory definition, on the other hand, supposes a further insight into the objects to which language refers. The name, circle, is defined as a perfectly round plane curve, as the name, straight line, is defined as a line lying evenly between its extremes. But ~~xxx~~ when one goes on to affirm that all radii in a circle are equal or that all right angles are equal, one no longer is talking merely of names. One is making assertions about the objects which names denote.

2.7 Primitive Terms.

A seventh observation adds a note on the old puzzle of primitive terms.

Every definition presupposes other terms. If these can be defined, their definitions will presuppose still other terms. But one cannot regress to infinity. Hence, either definition is based on undefined terms or else terms are defined in a circle so that each virtually defines itself.

Fortunately, we are under no necessity of accepting the argument's supposition. Definitions do not occur in a private vacuum of their own. They emerge in solidarity with experiences, images, questions, and insights. It is true enough that every definition involves several terms, but it is also true that no insight can be expressed by a single term, and it is not true that every insight presupposes previous insights.

Let us say, then, that for every basic insight there is a circle of terms and relations, such that the terms fix the relations, the relations fix the terms, and the insight fixes both. If one grasps the necessary and sufficient conditions for the perfect roundness of this imagined plane curve, then one grasps not only the circle but also the point, the line, the plane, the circumference, the radii, and equality. All the concepts tumble out together, because all are needed to express adequately a single insight. All are coherent, for coherence ~~hasx~~ basically means that all hang together from a single insight.

Again, there can be a set of basic insights. Such is the set underlying Euclidean geometry. ~~Theyxgenerate~~ Because the set of insights is coherent, they generate a set of coherent definitions. Because different objects of definition are composed of similar elements, such terms as point, line, surface, angle keep recurring in distinct definitions. Thus, Euclid begins his exposition from a set of images, a set of insights, and a set of definitions; some of his definitions are merely nominal; some are explanatory; some are derived, partly from nominally and partly from explanatorily defined terms.

2.8 Implicit Definition

A final observation introduces the notion of implicit definition.

D. Hilbert has worked out Foundations of Geometry that satisfy contemporary logicians. One of his important devices is known as implicit definition. Thus, the meaning of both point and straight line is fixed by the relation that two and only two points determine a straight line.

In terms of the foregoing analysis, one may say that implicit definition consists in explanatory definition without nominal definition. It consists in explanatory definition, for the relation that two points determine a straight line is a postulational element such as the equality of all radii in a circle. It omits nominal definition, for one cannot restrict the meaning of point to the Euclidean meaning of position without magnitude. An ordered pair of numbers satisfies Hilbert's implicit definition of a point, for two such pairs determine a straight line. Similarly, a first degree equation satisfies Hilbert's implicit definition of a straight line, for such an equation is determined by two ordered pairs of numbers.

The significance of implicit definition is its complete generality. The omission of nominal definitions is the omission of a restriction to the objects which, in the first instance, one happens to be thinking about. The exclusive use of explanatory ~~elements~~ or postulational elements concentrates attention upon the ~~p~~ set of relationships in which the whole of scientific significance is contained.

3. Higher Viewpoints. The next significant step to be taken in working out the nature of insight is to analyze development. Single insights occur either in isolation or in related fields. In the latter case they combine, ~~coalesce~~ cluster, coalesce, into the mastery of a subject; they ground sets of definitions, postulates deductions; they admit applications to enormous ranges of instances. But the matter does not end there. Still further insights arise. The short-comings of the previous position become recognized. New definitions and postulates are devised. A new and larger field of deductions is set up. Broader and more accurate applications become possible. Such a complex shift in the whole structure of insights, definitions, postulates, ~~and~~ deductions, and applications may be referred to very briefly as the emergence of a higher viewpoint. Our question is, Just what happens?

Taking our clue from Descartes' insistence on understanding simple things, we select as our pilot instance the transition from arithmetic to elementary algebra. Moreover, ~~lest some mathematician suppose that~~ to guard against possible misinterpretations, let us say that by arithmetic is meant a subject studied in grade school and that by elementary algebra is meant a subject studied in high school.

3.1 Positive Integers. A first step is to offer some definition of the positive integers, 1, 2, 3, 4,....

Let us suppose an indefinite multitude of instances of "one." They may be anything anyone pleases from sheep to instances of the act of counting or ordering.

Further, let us suppose as too familiar to be defined the notions of "one," "plus," and "equals."

~~Then, there is an infinite series of successive definitions for the infinite series of positive integers, namely, two is one more than one, three is one more than two, four is one more than three, etc., etc., or alternatively, the second is the~~

Then, there is an infinite series of definitions for the infinite series of positive integers, and it may be indicated symbolically by the following.

1 + 1 = 2
2 + 1 = 3
3 + 1 = 4
&c., &c., &c.

This symbolic indication may be interpreted in any of a variety of manners. It means one plus one equals two, or two is one more than one, or the second is the next after the first, or even the relations between classes of groups each with one, or two, or three, &c., members. As the acute reader will see, the one important element in the above series of definitions is the &c., &c., &c. Without it, the positive integers cannot be defined; for they are an indefinitely great multitude; and it is only in so far as some such gesture as &c., &c., &c., is really significant, that an infinite series of definitions can occur. What, then, does the &c., &c., mean? It means that an insight should have occurred. If one has had the relevant insight, if one has caught on, if one sees how the defining can go on indefinitely, no more need be said. If one has not caught on, then the poor teacher has to labor in his apostolate of the obvious. For in defining the positive integers there is no alternative to insight.

Incidentally, it may not be amiss to recall what already has been remarked, namely, that a single insight is expressed in many concepts. In the present instance, a single insight grounds an infinity of concepts.

3.2 Addition Tables. A second step will consist in making somewhat more precise the familiar notion of equality. Let us say that when equals are added to equals, the results are equal; that one is equal to one; and that therefore an ~~addition~~ infinite series of addition tables can be constructed.

The table for adding 2 is constructed by adding one to each side of the equations that define the positive integers. Thus,

From the table	$2 + 1$	$= 3$
Adding 1	$2 + 1 + 1$	$= 3 + 1$
Hence, from the table	$2 + 2$	$= 4$

In like manner the whole table for adding 2 can be constructed. From this table, once it is constructed, there can be constructed a table for adding 3. From that table it will be possible to construct a table for adding 4. &c., &c., &c., which again means that an insight should have occurred.

Thus, from the definitions of the ~~postive~~ positive integers and the postulate about adding equals to equals, there follows an indefinitely great deductive expansion.

3.3 The Homogeneous Expansion. A third step will be to venture into a homogeneous expansion. The familiar notion of addition is to be complemented by such further notions as multiplication, powers, subtraction, division, and roots. This development, however, is to be homogeneou[s] and by that is meant that no change is to be involved in the notions already employed.

Thus, multiplication is to mean adding a number to itself so many times, so that five by three will mean the additio[n] of three five's. Similarly, powers are to mean that a number is multiplied by itself so many times, so that five to the third will mean five multiplied by five with the result multiplied again by five. On the other hand, subtraction, division, and roots will mean the inverse operations that bring one back to the starting point.

By a few insights, that need not be indicated, it will be seen that tables for multiplication and for powers can be constructed from the addition tables. Similarly, tables for subtraction, division, and roots can be constructed from the tables for addition, multiplication, and powers.

The homogeneous expansion constitutes a vast extension of the initial deductive expansion. It consists in introducing new operations. Its characteristic is that the new operations involve no modification of the old.

The Need of a Higher Viewpoint.

3.4 A fourth step will be the discovery of the need of a higher viewpoint. This arises when the inverse operations are allowed full generality, when they are not restricted to bringing one back to one's starting point. Then, subtraction reveals the possibility of negative numbers, division reveals the possibility of fractions, roots reveal the possibility of surds. Further, there arise questions about the meaning of operations. What is multiplication when one multiplies negative numbers or fractions or surds? What is subtraction when one subtracts a negative number? &c., &c., &c. Indeed, even the meaning of "one" and of "equals" becomes confused, for there are recurring decimals and it can be shown that point nine recurring is equal to one.

Let	X	=	$0.\overline{9}$
then	$10X$	=	$9.\overline{9}$
hence	$9X$	=	9
and so	X	=	1

Formulation of the Higher Viewpoint.

3.5 A fifth step will be to formulate a higher viewpoint. Distinguish 1) rules, 2) operations, and 3) numbers.

Let numbers be defined implicitly by operations, so that the result of any operation will be a number and any number can be the result of an operation.

Let operations be defined implicitly by rules, so that what is done in accord with rules is an operation.

The trick will be to obtain the rules that fix the operations which fix the numbers.

The emergence of the higher viewpoint is the performance of this trick. It consists in an insight that 1) arises upon the operations performed according to the old rules and 2) is expressed in the formulation of the new rules.

Let me explain. From the image of a cart-wheel we proceeded by insight to the definition of the circle. But, while the cart-wheel was imagined, the circle consists of points and lines neither of which can be imagined. Between the cart-wheel and the circle there is an approximation but only an approximation. Now, the transition from arithmetic to elementary algebra is the same sort of thing. For an image of the cart-wheel one substitutes the image of what may be named "doing arithmetic"; it is a large, dynamic, virtual image that includes writing down, adding, multiplying, subtracting, dividing numbers in accord with the precepts of the homogeneous expansion. Not all of this image will be present at once, but any part of it can be present and, when one is on the alert, any part that happens to be relevant will pop into view. In this large and virtual image, then, there is to be grasped a new set of rules governing operation. The new rules will not be exactly the same as the old rules. They will be more symmetrical. They will be more exact. They will be more general. In brief, they will differ from the old much as the highly exact and symmetrical circle differs from the cart-wheel.

What are the new rules? ~~What did algebra greet you with in high school? There were rules of signs. There were rules for fractions. There were rules for~~ equations. These

What are the new rules? In high school the rules for fractions were generalized; rules for signs were introduced; rules for equations and for indices were worked out. Their effect was to redefine the notions of addition, multiplication, powers, subtraction, division, and roots; and the effect of the redefinitions of the operations was that numbers were generated, not merely by addition, but by any of the operations.

Successive Higher Viewpoints.

3.6 The reader familiar with group theory will be aware that the definition of operations by rules and of numbers or, more generally, symbols by operations is a procedure that penetrates deeply into the nature of mathematics. But there is a further aspect to the matter, and it has to do with the ~~development of mathematics~~ gradual development by which one advances through intermediate stages from elementary to higher mathematics. The logical analyst can leap from the positive integers to group theory, but one cannot learn mathematics in that simple fashion. On the contrary, one has to perform, over and over, the same type of transition as occurs in advancing from arithmetic to elementary algebra.

At each stage of the process there exists a set of rules that govern operations which result in numbers. To each stage there corresponds a symbolic image of doing arithmetic, doing algebra, doing calculus. In each successive image there is the potentiality of grasping by insight a higher set of rules that will govern the operations and by them elicit the numbers or symbols of the next stage. Only in so far as a man makes his slow progress up that escalator does he become a technically competent mathematician. Without it, he may acquire a rough idea of what mathematics is about; but he will never be a master, perfectly aware of the precise meaning and the exact implications of every symbol and operation.

The Significance of Symbolism.

3.7 The analysis also reveals the importance of an apt symbolism.

There is no doubt that, though symbols are signs chosen by convention, still some ~~symbols~~ choices are highly fruitful while others are not. It is easy enough to take the square root of 1764. It is another matter to take the square root of MDCCLXIV. The development of the calculus is easily designated in using Leibniz' symbol, dy/dx, for the differential coefficient; Newton's symbol, on the other hand, can be used only in a few cases and, what is worse, it does not suggest the theorems that can be established.

Why is this so? It is because mathematical operation are not merely the logical expansion of conceptual premises. Image and question, insight and concepts, all combine. The function of the symbolism is to supply the relevant image, and the symbolism ~~is apt inasmuch as its patterns and the automatic habits of doing it~~ is apt inasmuch as its immanent patterns as well as the dynamic patterns of its manipulation run parallel to the rules and operation that have been grasped by insight and formulated in concepts.

The benefits of this parallelism are manifold. In the first place, the symbolism itself takes over a notable part of the solution of problems, for the symbols, complemented by habits that have become automatic, dictate what has to be done. Thus, a mathematician will work at a problem up to a point and then announce that the rest is mere routine. In the second place,

the symbolism constitutes a heuristic technique: the mathematician is not content to seek his unknowns; he names them; he assigns them symbols; he writes down in equations all their properties; he knows how many equations he will need; and when he has reached that number, he can say that the rest ~~is just routine~~ of the problem is just routine. In the third place, the symbolism offers clues, hints, suggestions. Just as the definition of the circle was approached from the clue of the equality of the spokes, so generally insights do not come to us in their full stature; we begin from little hints, from suspicions, from possibilities; we try them out; if they lead nowhere, we drop them; if they promise success, we push them for all they are worth. ~~Thus~~ But this can be done only if we chance upon the hints, the clues, the possibilities; and the effect of the apt symbolism is to reduce, if not entirely eliminate, this element ~~of chance. Descartes, it is said, invented analytic geometry, which proceeds by symbols and equations, to avoid the mere chance that governs the discovery of the "construction"~~ of chance. Here, of course, the classical example is analytic geometry. To solve a problem by Euclidean methods, one has to stumble upon the correct construction. To solve a problem analytically, one has only to manipulate the symbols.
In the fourth place, there is the highly significant notion of invariance. An apt symbolism will ~~gives mathematics~~ endow the pattern of a mathematical expression with the totality of its meaning. Whether or not one uses the Latin, Greek, or Hebrew alphabet, is a matter of no importance. The mathematical meaning of an expression resides in the distinction between constants and variables and in the signs or collocations that dictate operations of combining, multiplying, summing, differentiat integrating, and so forth. It follows that, as long as the symbolic pattern of a mathematical expression is unchanged, its mathematical meaning is unchanged. Further, it follows that if a symbolic pattern is unchanged by any substitutions of a determinate group, then the mathematical meaning of the pattern is independent of the meaning of the substitutions.
In the fifth place, as has already been mentioned, the symbolism appropriate to any stage of mathematical development provides the image in which may be grasped by insight the rules for the next stage.

There follow pp 59–74.
(end of elements)
& p 75. Heuristic Structures

4. Inverse Insight.

Besides direct insights, their clustering, and higher viewpoints, there exists the small but significant class of inverse insights. As direct, so also inverse insights presuppose a positive object that is presented by sense or represented by imagination. But while direct insight meets the spontaneous effort of intelligence to understand, inverse insight responds to a more subtle and critical attitude that distinguishes different degrees or levels or kinds of intelligibility. While direct insight grasps the point, or sees the solution, or comes to know the reason, inverse insight apprehends that in some fashion the point is that there is no point, or that the solution is to deny a solution, or that the reason is that the rationality of the real admits distinctions and qualifications. Finally, while the conceptual formulation of direct insight affirms a positive intelligibility though it may deny expected empirical elements, the conceptual formulation of an inverse insight affirms empirical elements only to deny an expected intelligibility.

Since the last phrase is crucial, let us attempt to elaborate it. By intelligibility is meant the content of a direct insight. It is the component that is absent from our knowledge when we do not understand and added to our knowledge inasmuch as we are understanding in the simple and straightforward manner described in the earlier sections of this chapter. Now such an intelligibility may be already reached or it may be merely expected. To deny intelligibility already reached is not the result of inverse insight; it is merely the correction of a previous direct insight, the acknowledgement of its shortcomings, the recognition that it leaves problems unsolved. But to deny an expected intelligibility is to run counter to the spontaneous anticipations of human intelligence; it is to find fault not with answers but with questions. In a demonstrative science it is to prove that a question of a given type cannot be answered. In an empirical science it is to put forward a successful hypothesis or theory that assumes that certain questions mistakenly are supposed to require an answer. Finally, the occurrence of an inverse insight is not established by the mere presence of negative concepts: thus, "not-red," "position without magnitude," "non-occurrence" exclude respectively "red," "magnitude," "occurrence"; but the latter terms refer to empirical components in our knowledge and not to the possibilities and necessities, the unifications and relations, that constitute the intelligibility known in direct insight.

While the general notion of inverse insight is fairly simple and obvious, I have been at some pains in presenting its characteristics because it is not too easy to set forth illustrations to the satisfaction of different groups of readers. Moreover, communication and discussion take place through concepts, but all insight lies behind the conceptual scene. Hence, while there is always the danger that a reader will attend to the concepts rather than the underlying insight, this danger is augmented considerably when the point to be grasped by insight is merely that there is no point. To make matters worse, inverse insights occur only in the context of far larger developments of human thought. A statement of their content has to call upon the later systems that positively exploited their negative contribution. The very success of such later systems tends to engender a routine that eliminates the more spontaneous anticipations of intelligence and then, to establish a key feature of an inverse insight, it may be necessary to appeal to the often ambiguous witness of history. In the midst of such complexity it very easily can happen that a reader's spontaneous expectation of an intelligibility to be reached should outweigh mere verbal admonitions to the contrary and, when that happens occurs, illustrations of inverse insight can become very obscure indeed. Accordingly, while there is nothing difficult about the examples to follow, I have thought it wise to indulge in an apostolate of the obvious.

As a first example of inverse insight we shall take what the ancients named incommensurable magnitudes and the moderns call irrational numbers. In both cases there is a positive object indicated by the terms, "magnitude," "number." In both cases there is a negative element indicated by the epithets, "incommensurable," "irrational." Finally, in both cases the negation bears on the spontaneous anticipations of human intelligence. "Incommensurable" denies the possibility of applying to certain magnitudes some type of measurement and Aristotle viewed this denial as *prima facie* a matter of high surprise. Even more emphatically "irrational" denies a correspondence between certain numbers and human reason.

To indicate the relevant insight, let us ask why a surd is a surd. Essentially the question is parallel to the earlier question, Why is a cart-wheel round? But while the earlier answer revealed an intelligibility immanent in the wheel, the present answer consists in showing that a surd cannot possess the intelligibility one would expect it to have.

Thus, the square root of two is some magnitude greater than unity and less than two. One would expect it to be some improper fraction, say *m n* , where *m* and *n* are positive integers and, by the removal of all common factors, *m* may always be made prime to *n* . Moreover, were this expectation correct, then the diagonal and the side of a square would be respectively *m* times and *n* times some common unit of length. However, so far from being correct, the expectation leads to a contradiction. For if $\sqrt{2} = \underline{m}/\underline{n}$, then $2 = \underline{m}^2/\underline{n}^2$. But if $\underline{m}$ is prime to $\underline{n}$, then $\underline{m}^2$ is prime to $\underline{n}^2$; and in that case $\underline{m}^2/\underline{n}^2$ cannot be equal to 2 or, indeed, to any integer. The argument is easily generalized and so it appears that a surd is a surd because it is not the rational fraction that ~~intelligib~~ intelligence anticipates it to be.

A second example of inverse insight is the non-countable multitude. There is a positive object, "multitude." There is a negative determination, "non-countable." Moreover, when "countable" is taken so broadly that all integers, all rational numbers, even all real algebraic numbers (*) demonstrably are countable multitudes, when further it can be shown that to remove a countable multitude from a non-countable multitude leaves a non-countable multitude, one spontaneously anticipates that the numbers between zero and unity must be a countable multitude. In fact, it can be shown that the infinite decimals are a non-countable multitude, so that ~~the rational and~~ the algebraic ~~irrational~~ fractions from zero to unity must be a negligible portion of the real numbers in that interval. (*)

(*) Algebraic numbers are the roots of algebraic equations with integral coefficients. For a generous exposition of the topic ~~nns~~ and its paradoxes see A. Fraenkel, Abstract Set Theory, Amsterdam 1953, pp. 43 - 75. For applications to the continuum, see pp. 212 ff.

For a third example we turn to empirical science and consider the surprising part of Newton's first law of motion, namely, that a body continues in its existing state of uniform motion in a straight line unless that state is changed by

In this statement and its content it is not too difficult to discern the three characteristics of the formulation of an inverse insight. For there is the positive object: a body continues to move at a uniform rate in a straight line. There is a negation: the continuance of the constant velocity depends not on the action of external force but on the absence of such action; for only as long as there is no acceleration, does the velocity remain constant; and at the moment the sum of the external forces differs from zero, there arises an acceleration. Finally, this negation of external force runs counter to the spontaneous anticipations of human intelligence, for spontaneously one thinks of uniform motion not as of a state like rest but as of a change that requires an external cause.

However, some readers may wish to refine on the issue. They will agree that the necessity of an external cause had been stressed by the Aristotelian theory of celestial movements, of projectiles, and of motion in a vacuum. But they will add that the Aristotelian view had been contradicted at least from the time of John Philoponus. On this contrary view projectiles were kept in motion not by any external force but by some internal principle or power or property or quality or other immanent ground. Finally, they will ask whether it is quite certain that Newton did not appeal to some innate power of ~~get~~ matter to account for the continuance of inertial states.

Now, clearly, Newtonian exegesis is not our present business. All we have to say is that inverse insight is not illustrated when explanation by external force is replaced by explanation in terms of some immanent power or property. For in that case there is merely the correction of an earlier direct insight by a latter direct insight and, while the spontaneous anticipations of human intelligence are blocked in one direction, they are given an outlet in another.

Still for purposes of illustration it may be permissible to block this second outlet without reopening the first. No doubt, when an external mover or force is denied, one may spontaneously think that there must be some innate quality that provides the real explanation. But while the assertion of an external mover or force can be tested experimentally, the assertion of some innate quality, of some *vis materiae insita*, can hardly be regarded as a scientific statement. If one affirms that, when acceleration is zero, then the sum of the relevant external forces is also zero, one's affirmation admits the ordinary tests. But if one goes on to add that the innate qualities of matter render the action of external forces superfluous, one is very likely to be reminded that scientists do not appeal to occult causes.

Now if this remonstrance is regarded as peremptory, we arrive at an example of inverse insight. There is the positive object of inquiry: bodies continue in their existing states of uniform motion. There is the negation: the continuance of uniform motion is not to be explained by any appeal to external forces. Finally, this negation is regarded as definitive for science, for science refuses to extrapolate from known laws to ulterior explanations in terms of vague qualities, properties, powers, and the like.

A fourth example of inverse insight may be derived from the basic postulate of the Special Theory of Relativity. The postulate itself is that the mathematical expression of physical principles and laws is invariant under inertial transformations. To reach our illustration we have only to grasp the concrete meanin of the postulate whenever it is invoked by a physicist engaged in understanding any set of physical data.

For then the positive object of inquiry consists in the data inasmuch as they are considered 1) as referred to initial axes of coordinates, say K, and 2) as referred to other axes, say K', moving with a constant velocity relative to the axes, K.

The negative element in the conception of the positive object is indicated by the word, "invariant." It means that the transformation from one set of axes to another does not lead to any modification in the form of the mathematical expression of the appropriate physical principles and laws. But when the form of the mathematical expression undergoes no change, there is no change in the intelligibility that is expressed mathematically. When there is no change in the intelligibility, there is no change in the act of understanding that grasps the intelligibility and expresses it mathematically. Accordingly, the concrete meaning of the postulate is that, though there is a difference in the spatio-temporal standpoint from which the data are considered, still there is no difference in the act of understanding the data, no difference in the general intelligibility grasped in the data, and no difference in the form of the mathematical expression of the intelligibility.

Finally, it is quite common for there to exist differences either in data or in spatio-temporal standpoint without any corresponding difference in the act of understanding. But in most of such cases there is no occasion for an inverse insight ~~they~~ since, while the empirical difference is assigned no intelligible counterpart, still no one expects that really there must be an intelligible counterpart. Thus, there is a notable empirical difference between large and small circles, yet no one expects different definitions of large circles and of small circles or different theorems to establish the different properties of large and small circles. However, while similar instances are very numerous, the invariance

~~aspect~~ postulated by Special Relativity is not among them. For that invariance implies a drastic revision of ordinary notions of space and of time, and against any such revision the spontaneous anticipations of human intelligence vigorously rebel.

Hence, to recapitulate the main point, when the basic postulate of Special Relativity is interpreted concretely in terms of 1) the data physicists consider, 2) the insights they enjoy, and 3) the form of the mathematical expression of the principles and laws reached by the insights, there arises the following explanatory syllogism:

When there is no difference in a physicist's insights, there should be no difference in the form of the mathematical expression of physical principles and laws.

But when an inertial transformation occurs, there is no difference in a physicist's insights.

Therefore, when an inertial transformation occurs, there should be no difference in the form of the mathematical expression of physical principles and laws.

The major premise postulates a correspondence between the insights of physicists and the form of the mathematical expression of physical principles and laws; in other words, it requires that the content of acts of understanding be reflected faithfully by the form of mathematical expressions. The minor premise contains our inverse insight: it denies a difference in insight that corresponds to the difference of an inertial transformation; in other words, it asserts for the whole of physics the defect of intelligibility in constant velocity that Newton asserted for mechanics by ~~in~~ his first law of motion. The conclusion, finally, is true if the premises are true but, while the major premise may be regarded as a mere methodological rule, the minor premise is an assertion of empirical science and can be established only through the method of hypothesis and verification.

In conclusion, let us recall a point already mentioned. An inverse insight finds its expression only in some concomitant positive content. So the defect of intelligibility in ~~content~~ constant velocity has been formulated in a whole series of different contexts. In the context of Eleatic philosophy Zeno's paradoxes ~~lead~~ led to a denial of the fact of motion. In the context of his philosophy of being Aristotle pronounced motion real yet regarded it as an incomplete entity, an infra-categorial object. In the context of mathematical mechanics Newton asserted a principle of inertia. In the context of Clerk-Maxwell's equations for the electromagnetic field Lorentz worked out the conditions under which the equations would remain invariant under inertial transformations, Fitzgerald explained Lorentz'~~s~~ success by supposing that bodies contracted along the direction of motion, Einstein found a ~~more~~ no less general explanation in problems of synchronization and raised the issue to the methodological level of the transformation ~~of~~ properties of the mathematical expression of physical principles and laws, finally Minkowski systematized Einstein's position by introducing the four-dimensional manifold. No doubt, it would be a mistake to suppose that the same inverse insight was operative from Zeno to Special Relativity. But throughout there is a denial of intelligibility to local motion and, while the successive contexts differ notably in content and in value, at least they point in the same direction and they illustrate the dependence of inverse insight on concomitant direct insights.

5. The Empirical Residue.

If inverse insights are relatively rare, they are far from being unimportant. Not only do they eliminate mistaken questions but also they seem regularly to be connected with ideas or principles or methods or techniques of quite exceptional significance. From the oddities of the mathematical continuum through the notions of correlation and limit there arises the brilliance of continuous functions and of the infinitesimal calculus. Similarly the lack of intelligibility in constant velocity is linked with scientific achievements of the first order: the principle of inertia made it possible to conceive dynamics not as a theory of motions but as an enormously more compact and more powerful theory of accelerations; and the invariance of physical principles and laws under inertial transformations not only is an extremely neat idea but also has kept revealing its fruitfulness for the past fifty years.

To explore this significance, then, let us introduce the notion of an empirical residue that 1) consists in positive empirical data, 2) is to be denied any immanent intelligibility of its own, and 3) is connected ~~with~~ with some compensating higher intelligibility of ~~its own~~ notable importance. In clarification of the first characteristic one may note that, inasmuch as a vacuum is merely an absence of data, it cannot be part of the empirical residue. In clarification of the second it is to be remembered that a denial of immanent intelligibility is not a denial of experience or description. Not only are elements in the empirical residue given positively but also they are pointed out, conceived, named, considered, discussed, and affirmed or denied. But though they are no less given than color or sound or heat, though they may be thought about no less accurately and talked about no less fluently, still they are not objects of any direct insight and so they cannot be explained by transverse waves or longitudinal waves or molecular motion or any other theoretical construct that might be thought more apposite. Finally, in clarification of the third characteristic it is to be noted that inverse insight and the empirical residue are not exact correlatives. For inverse insight was not characterized by a connection with ideas, principles, methods, or techniques of exceptional significance. Again, the empirical residue has not been characterized by the spontaneity of the questions for intelligence that are to be met by a denial of intelligibility.

This difference not only makes the empirical residue a broader category than inverse insight but also renders a discussion of it more difficult. For a great part of the difficulty in discovering the further positive aspects of experience that are to be denied intelligibility is that no one supposes them to possess intelligibility.

Thus, particular places and particular times pertain to the empirical residue. They are positive aspects of experience. Each differs from every other. But because no one ever asks why one place is not another or why one time is not another, people are apt to be puzzled when the question is put, to imagine that something different from such obvious foolishness must be meant, and to experience a variety of fictitious difficulties before arriving at the simple conclusion that 1) particular places and particular times differ as a matter of fact and 2) there is no immanent intelligibility to be grasped by direct insight into that fact.

For example, one will begin by saying that obviously the position, A, differs from the position, B, because of the distance, AB, that separates them. But take three equidistant positions, A, B, C. Why are the distances, AB, BC, CA, different? One would be in a vicious circle if one doubled back and explained the difference of the distances by the difference of the positions. One cannot say that the distances differ in length for they are equal in length. But one may say that the distances differ because the directions differ? Still, why do the directions differ? And why are equal and parallel distances different distances? Now, perhaps, it will be urged that we are going too far, that some ~~differ~~ difference must be acknowledged as primitive, that everything cannot be explained. Quite so, but there is a corollary to be added. For what is primitive is not the content of some primitive insight but the content of some primitive experience to which no insight corresponds. Were it the content of some primitive insight, there would not be the conspicuous absence of a clear-headed explanation. But because the difference of particular places and the difference of particular times are given prior to any questioning and prior to any insight, because these given differences cannot be matched by any insights that explain why places differ and times ~~i~~ differ, there has to be introduced the category of the empirical residue.

However, one may not surrender yet. For particular places and particular times can be united by reference frames; the frames can be employed to distinguish and designate every place and every time; and evidently such constructions are eminently intelligent and ~~obviously~~ eminently intelligible. Now, no doubt, reference frames are objects of direct insight, but what is grasped by that insight is an ordering of differences that are not explained by the order but merely presupposed. So it is that different geometries grasped by different insights offer different intelligible orders for the differences in place or time that all equally presuppose and, quite correctly, none attempt to explain.

Stet ~~There is a further aspect to the matter. Because particlarp particular places and particular times possess no immanent~~

But there is a further aspect to the matter. Because particular places and particular times possess no immanent [71] intelligibility of their own, they cannot involve any modification of the intelligibility of anything else. It is not mere difference in place but something different ~~at~~ in the place, that gives rise to different observations or different experimental results in different places. Similarly, it is not mere difference in time but something different at that time that gives rise to different observations or different experimental results at different times. Moreover, were that not so, every place and every time would have its own physics, its own chemistry, its own biology; and as a science can not be worked out instantaneously at a ~~mmmm~~ single place, there would be no physics, no chemistry, and no biology. Conversely, because particular places and particular times pertain to the empirical residue, there exists the powerful technique of scientific collaboration; scientists of every place and every time can pool their results in a common fund and there is no discrimination against any result merely because of the place or merely because of the time of its origin.

Even more fundamental ~~that~~ than scientific collaboration is scientific generalization. When chemists have mastered all of the elements, their isotopes, and their compounds, they may forget to be grateful that they do not have to discover different explanations for each of the hydrogen atoms which, it seems, make up about fifty-five per cent of the matter of our universe. But at least the fact that such a myriad of explanations is not needed is very relevant to our purpose. Every chemical element and every compound differs from every other kind of element or compound and all the differences have to be explained. Every hydrogen atom differs from every other hydrogen atom and no explanation is needed. Clearly, we have to do with another aspect of the empirical residue and, no less clearly, this aspect is coupled with the most powerful of all scientific techniques, generalization.

However, this issue has been bo[u]ted about by philosophers ever since the Platonists explained the universality of mathematical and scientific knowledge by postulating eternal and immutable Forms or Ideas only to find themselves embarrassed by the fact that a single, eternal, immutable One could hardly ground the universal statement that one and one are two or, again, that a single, eternal, immutable Triangle would not suffice for theorems on triangles similar in all respects. So there arose, it seems, the philosophic problem of merely numerical difference and, connected with it, there have been formulated cognitional theories based on a doctrine of abstraction. Accordingly, we are constrained to say something on these issues and, lest we appear to be attempting to dilute water, we shall do so as briefly as possible.

The assertion, then, of merely numerical difference involves two elements. On the theoretical side it is the claim that, when any set of data have been explained completely, another set of data similar in all respects would not call for a different explanation. On the factual side it is the claim that, when any set of data has been explained completely, only an exhaustive tour of inspection could establish that there does not exist another set of data similar in all respects.

The basis of the theoretical contention is that, just as the same act of understanding is repeated when the same set of data is apprehended a second time, so also the same act of understanding is repeated when one apprehends a second set of data that is similar to a first in all respects. Thus, the physicist offers different explanations for "red" and "blue"; he offers different explanations for different shades of "red"; and he would discern no sense in the proposal that he should try to find as many different explanations as there are different instances of exactly the same shade of exactly the same color.

The factual contention is more complex. It is not an assertion that there exist different sets of data similar in all respects. It is not a denial of unique instances, i.e., of instances that are to be explained in a manner in which no other instance in the universe is to be explained. It is not even a denial that every individual in the universe is a unique instance. On the contrary, the relevant fact lies in the nature of the explanations that are applicable to our universe. It is to the effect that all such explanations are made up of general or universal elements and that, while these general or universal elements may be combined in such a manner that every individual is explained by a different combination of elements, still such a combination is an explanation of a singular combination of common properties and not an explanation of individuality. For if the individuality of the individual were explained, it would be meaningless to suppose that some other individual might be understood in exactly the same fashion. On the other hand, because the individuality of the individual is not explained, it is only an exhaustive tour of inspection that can settle whether or not there exists another individual similar in all respects. Hence, even if there were reached a single comprehensive theory of evolution that explained and explained differently every instance of life on this planet, still in strict logic we should have to inspect all other planets before we could be absolutely certain that in fact there did not exist another instance of evolution similar in all respects.

In brief, individuals differ, but the ultimate difference in our universe is a matter of fact to which there corresponds nothing to be grasped by direct insight. Moreover, as scientific collaboration rests on the empirically residual difference of particular places and of particular times, so scientific generalisation rests on the empirically residual difference between individuals of the same class. Just what the lowest class is, has to be discovered by scientific advance in direct insight. Even if it should prove that in some sense there are as many classes as individuals, still we can know at once that that sense is not that the individuality of individuals is understood but merely that singular combinations of universal explanatory elements may be set in correspondence with singular combinations of common properties or aspects in each individual. For the content grasped in insight can be embodied no less in imagination than in sense; and whether there is more than one instance in sense, can be settled only by an empirical tour of inspection.

Later we shall direct attention to further aspects of the empirical residue, for there exists a statistical method that rests on the empirically residual character of coincidental aggregates of events, and there is a dialectical method that is necessitated by the lack of intelligibility in man's unintelligent opinions, choices, and conduct. But perhaps enough has been said for the general notion to be clear, and so we turn to the allied topic of abstraction.

Properly, then, abstraction is not a matter of apprehending a sensible or imaginative Gestalt; it is not a matter of employing common names just as it is not a matter of using other tools; finally, it is not even a matter of attending to one question at a time and, meanwhile, holding other questions in abeyance. Properly, to abstract is to grasp the essential and to disregard the incidental, to see what is significant and ~~sed~~ set aside the irrelevant, to recognize the important as important and the negligible as negligible. Moreover, when it as asked what is essential or significant or important and what is incidental, irrelevant, negligible, the answer must be twofold. For abstraction is the selectivity of intelligence, and ~~intelligence~~ intelligence may be considered either in some given stage of development or at the term of development when some science or group of sciences has been mastered completely.

Hence, relative to any given insight or cluster of insights, the essential, significant, important consists 1) in the set of aspects in the data necessary for the occurrence of the insight or insights or 2) in the set of related concepts necessary for the expression of the ~~basic~~ insight or insights. On the other hand, the incidental, irrelevant, negligible consists 1) in other concomitant aspects of the data that do not fall under the insight or insights or 2) in the set of concepts that correspond to the merely concomitant aspects of the data. Again, relative to the full development of a science or group of allied sciences, the essential, significant, important consists 1) in the aspects of the data that are necessary for the occurrence of all insights in the appropriate range or 2) in the set of related concepts that express all the insights of the science or sciences. On the other hand, the incidental, irrelevant, negligible consists in the empirical residue that, since it possesses no immanent intelligibility of its own, is left over without explanation even when a science or group of sciences reaches full development.

Finally, to conclude this chapter on the Elements of Insight, let us indicate briefly what is essential, significant, important in its contents and, on the other hand, what is incidental, irrelevant, negligible. What alone is essential is insight into insight. Hence, the incidental includes 1) the particular insights chosen as examples, 2) the formulation of these insights, and 3) the images evoked by the formulation. It follows that for the story of Archimedes the reader will profitably substitute some less resounding yet more helpful experience of his own. Instead of the definition of the circle he can take any other intelligently performed act of defining and ask why the performance is, not safe, not accurate, not the accepted terminology, but a creative stroke of insight. Instead of the transition from elementary arithmetic to elementary algebra one may review the process from Euclidean to Riemannian geometry. Instead of asking why surds are surds, one can ask why transcendental numbers are transcendental. Similarly, one can ask whether the principle of inertia implies that Newton's laws are invariant under inertial transformations, what inspired Lorentz to suppose that the electromagnetic equations should be invariant under inertial transformations, whether an inverse insight accounts for the basic postulate of General Relativity, whether the differences of particular places or particular times are the same aspect of the empirical residue as the differences of completely similar hydrogen atoms. For just as in any subject one comes to master the essentials by varying the incidentals, so one reaches familiarity with the notion of insight by modifying the illustrations and discovering for oneself and in one's own terms the point that another attempts to put in terms he happens to think ~~we~~ will convey the idea to a ~~[illegible]~~ probably non-existent average reader.