Inverse Insight.

4. So far we have been asking questions that can be answered. How can one tell whether a crown is made of pure gold without melting it down? What accounts for a wheel being round? What is arithmetic and how does one go on to algebra? In each case, there is an appropriate image or set of images that, under the stress of inquiry, result in an insight that expresses itself in some formulation called the answer.

Now attention has to be directed to a quite different case. There is the question. There is the answer. But the answer consists in showing the question to be misconceived, and it is grounded in an insight that grasps why the question, as conceived, cannot be answered.

4.1 Surds. How big is the square root of two? Clearly, it is greater than one, for the square of one is one; and it is less than two, for the square of two is four. It would seem, then, that it is some improper fraction lying between one and two.

Now an improper fraction is the quotient of some positive integer divided by some other, smaller positive integer. Moreover, it is always possible to reduce such a fraction to its lowest terms by removing all common factors. Let us suppose then that

$$\sqrt{2} = M/N$$

where M and N are positive integers with no common factors. Multiplying across by N and squaring, one obtains

$$2N^2 = M^2$$

It follows that M must be an even number and so twice, say, P. Substituting and dividing by two, one obtains

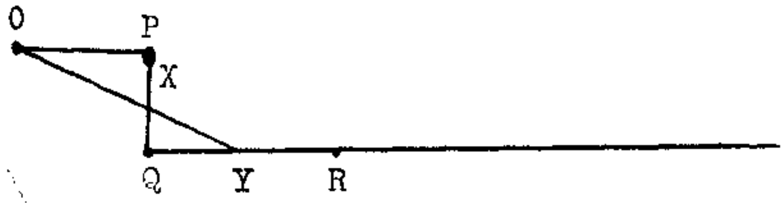
$$N^2 = 2P^2$$

so that N also must be an even number, which contradicts the assumption that all common factors were eliminated. It follows that there is no "rational" fraction, M/N, that is equal to the square root of two. Moreover, since any recurring decimal can be reduced to such a fraction, there is no recurring decimal equal to the square root of two. However, one can apply to 2 the ordinary method for taking the square root, and so it remains that the square root of two will be an infinite, non-recurring decimal. Finally, the foregoing argument can be generalized and applied to any surd. Thus, if

$$3N^2 = M^2$$

then 3 must be a factor of M, so that M can be replaced by 3P, whence it will follow that 3 must be a factor of N.

4.2 Non-Countable Infinity. Again, to raise another, similar question, How many points are there in a straight line one inch long? Clearly, the number must be very large, for a point is position without magnitude. But, at least, one would be inclined to say that there are twice as many points in a straight line two inches long as in a straight line one inch long. Still, that would be erroneous, as appears from the following construction. Let the straight line, PQ, be perpendicular to the straight lines, OP and QR. Let QR be twice as long as PQ. And let OXY be a straight line cutting PQ in X and QR in Y.




Then, from the construction, it is clear that for every point, Y, in QR there is a corresponding, distinct point, X, in PQ. Indeed, this remains true if one produces QR to infinity in the direction of R. No matter where Y is taken on QR produced, there is always a corresponding and distinct point X in PQ. Hence, there are as many points in an inch of straight line as there are in two inches, or in a foot, or in a mile, or in as many light-years as you please.

However, we have not met the question. We have said there are <u>as many as...</u> We have not said <u>how many.</u> Accordingly, let us distinguish between the counted, the countable, and the non-countable. A set is counted when one say it contains N members where N is some positive integer. A set is countable when it ~~is~~ can be arranged in some determinate order that contains all its members ~~xxx~~ once each and only once; for then there can be established a one-to-one correspondence between the members of the set and the positive integers. ~~It is to be noted that, a countable set may be infinite~~ Finally, a set is non-countable when it is not possible to ~~s~~ establish a one-to-one correspondence between its members and the positive integers.

It is to be noted that by "countable" is not meant the possibility of finishing the counting. Thus, an ~~f~~ infinite series, such as

1/2, 1/4, 1/8, 1/16,....

is countable, for all its members lie in a determinate order and so can be placed in a one-to-one correspondence with the positive integers. Again, an infinite series of infinite series of elements is countable, for all its elements can be regarded as lying within a single determinate order. Thus, the reciprocals of the <u>nth</u> powers of the prime numbers form an infinite series of infinite series. Their elements can be arranged in a column of rows, thus,

~~1/2 1/4 1/8 1/16~~
~~1/9 1/27 1/81 1/243~~

| | | | | |
|---|---|---|---|---|
| 1/2 | 1/4 | 1/8 | 1/16 | .... |
| 1/3 | 1/9 | 1/27 | 1/81 | .... |
| 1/5 | 1/25 | 1/125 | 1/525 | .... |
| 1/7 | 1/49 | 1/343 | 1/2401 | .... |
| ........ | | | | |

and any column of rows can be counted in the following manner,

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 5 | 10 | 17 |
| 4 | 3 | 6 | 11 | etc. |
| 9 | 8 | 7 | 12 | |
| 16 | 15 | 14 | 13 | |

Thus, any infinite series of infinite series can be assigned the order of a single infinite series. It follows that an infinite series of infinite series of ~~xi~~ infinite series can be arranged in a column of rows and so can be assigned the order of a single infinite series. The theorem can be repeated indefinitely. Thus, consider the rational, proper fractions,

1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 1/6,....

From this infinite series there can be derived an infinite series of infinite series, for one can take, first, the square root of the lot, then, the cube root, then square the cube roots, then take the fourth root, then cube the fourth roots, etc. Now, as has been shown, this infinite series of infinite series can be arranged in a single series. Once this is done, one can use these new terms as powers to be applied to the rational proper fractions to derive a new infinite series of infinite series. This can be arranged in a single series, applied as powers to the rational proper fractions, yield a new infinite series of infinite series, etc., etc.

From the foregoing it is clear that any infinite set is countable, provided it is possible to assign some order to its members. It is also clear that a non-countable infinite set must contain such a ~~multipl~~ multitude of members in such a manner than ordering them is impossible. Such is the case with the points in a straight line. Thus, in the line, QR, it is impossible to pick any point, Q', that is nearest to Q; for however short QQ' may be, it contains as many points as there are in a line as long as you please. Nor is there any use trying to proceed by dividing the line. For if this could be done in an orderly fashion, then one would be appealing to an ordered series of all the numbers greater than zero and less than unity. But that range of numbers is a non-countable infinite set, for it cannot be arranged in a/~~column of rows~~. single order./ Suppose there were some single column containing all the infinite decimals. Then ~~consider the diagonals and proceed to construct infinite decimals~~ consider the diagonal. It is always possible to construct another infinite decimal that differs from the first infinite decimal by the digit in the first place, from the second by the digit in the second place, from the nth by the digit in the nth place. Therefore, the initial assumption is false. The column did not contain all the infinite decimals. There is, then, no single series that contains all the infinite decimals and so the infinite decimals are a non-countable infinite set.

Well, how many points are there in a straight line an inch long? There is no answer. They form a non-countable infinite set. They do so, because they cannot be placed in a single order and so cannot be correlated in a one-to-one correspondence with the positive integers.

Suggestions to Dennis - Nov 2/54.

Elements:

In 4.3

Replace "What is the mathematician's ..... between non-countable infinite sets"

by "What is a mathematician after in defining a continuous function? ~~and~~ Fundamentally he wants a correspondence between non-countable infinite sets"

Replace "Thus, consider ... if the values of $x$ are a non-countable infinite set"

by "Thus, consider the function, $y = x^2$. It is a function since, for every value of $x$ there is a corresponding value of $y$. We'll say it is fundamentally continuous in that the values of $x$ are a non-countable set.

Replace "Now as $x$ increases ..... square of $x$."

by "We'll suppose $x$ greater than one. As $x$ increases, $y$ must increase ~~more~~ more rapidly for it equals the square of $x$"

Replace: "Still, there are no points ..... along $y$."

by Still no points are omitted along $x$ (nor any along $y$)

Replace "Thus, if the limit of the ratio .... $dy/dx = 2x$."

by. So we have a new function, $2x$, derived from the original function, $x^2$, which measures, by a limit, the ratio of the increment in $y$ to the increment in $x$. This function, denoted by the symbol $dy/dx$, is called the "derivative function" or merely "the derivative".

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In §52 (of my corrected copy) In place of: "Next a function, $f(x)$ ..."

write: Next, a function $f(x)$ will be said to be fundamentally continuous in a range if.. etc as in text but replace in the paragraph "continuous function" by "fundamentally continuous function".

However, they can be placed in a one-to-one correspondence with other non-countable infinite sets. Thus, there are as many points in an inch as in a mile or a light-year or in as many light-years as you please. But that does not mean that there is some determinate number of points in an inch or in a mile. Much less does it mean that some smaller number is equal to a greater number. There just is no numbering, no counting. And there is no numbering or counting because there is no possibility of effecting an order, a system, an arrangement.

4.3 Functions and Limits. One might think that this exclusion of number and of order blocked the mathematician. In fact, it gives him a new lease of life. What is the mathematician's continuous function? In the elementary case it is a one-to-one correspondence between non-countable infinite sets. Moreover, since such a correspondence can be set up between an inch and a foot, or an inch and a ~~kk~~ mile, or an inch and a light-year, or any intermediate or still odder pair, since, visually ~~more remarkably~~, length is independent of the number of points, the mathematician can develop the infinitesimal calculus. But he does so, not by finding some order in the non-countable infinite set, but by developing a technique of getting around it. This technique is named proceeding to the limit.

Thus, consider the continuous function, $y = x^2$. It is a function if, for every value of x, there is a corresponding value of y. It is a continuous function. if the values of x are a non-countable infinite set, ~~and for every distinct value of x there is a distinct value of y~~.

Now as x increases, y must increase more rapidly, for it equals the square of x. Hence, visually, as one moves from point to point along x, one must move more rapidly from point to point along y. Moreover, the further one advances along x, the greater must be one's strides along y. Still, there are no points omitted along x and there are no points omitted along y.

What, then, is the ratio of the increment of y to the increment of x? Clearly, if x increases by some slight amount, h, y will increase by

$$(x + h)^2 - x^2 = 2xh + h^2$$

Hence the ratio of the corresponding increment of y to the increment, h, of x will be $(2x + h)$. The smaller the increment, h, the nearer is the ratio to 2x. In the limit, it is exactly 2x. Thus, if the limit of the ratio of the increment in y to the increment in x is denoted by the symbol, dy/dx, then besides the initial function, $y = x^2$, we have also the derivative function, $dy/dx = 2x$.

Now what is this business of proceeding to the limit? There is said to~~xx~~ be a limit, P, to a non-determined quantity, Q, if the difference of Q from P can be made smaller than any number one cares to assign. Thus, above, by making the increment, h, smaller and smaller, one can make the difference of $(2x + h)$ from 2x as small as we please. Still, this is only the conceptual formulation of the procedure of taking a limit. What is the underlying insight? What is the image that the insight presupposes?

Clearly enough the image will differ in different cases. Similarly, the insight will be reached in different manners. But the peculiarity of the insight is that it grasps, not that something is to some point, but that something is beside the point. No matter how small h is, there is a non-countable infinite set of values between 2x and (2x + h). They are non-countable because they defy arrangement, order, system. They exhibit a non-systematic aspect of continuous variables and continuous functions. But what one is trying to do in mathematics, is to reach the systematic. If that is all one wants, one can disregard the non-systematic. ~~If can leap over~~ One can leap over the non-countable infinity because it is without order if one's aim is to grasp just what admits order. Again, the ratio of the increment of y to the increment of x is any of a non-countable infinite set of values. But the limit of that ratio is unique. It can be determined systematically. It pertains to system.

4.4 Abstraction. When one comes to think of it, we have been doing this sort of thing all along. The principles of displacement and of specific gravity would not enable ~~to~~ Archimedes to determine that there was nothing but pure ~~unalloyed~~ gold in the crown; they would enable him to say merely that there was extremely little else. Again, the definition of the circle paid no attention to the size, the weight, the strength, the origin, the materials, the purpose of the cart-wheel; on the contrary, it went off to a realm of the non-imaginable where points have position without magnitude and lines ~~ix~~ have length without thickness. Finally, the transition from arithmetic to ~~algebra~~ consist/algebra did not/~~consisted~~ in paying closer attention to the things one might count by the positive integers; it consisted in deserting the good, common sense notion of adding and in developing a new notion that gave a meaning to adding negative numbers, multiplying fractions, and doing other things that have no prima facie meaning.

It is time, then, for us to reflect on certain general aspects of the process from image ~~thru~~ through insight to conceptions, and so we had best begin a new section.

The Empirical Residue.

5. The suppositions and conceptions resulting from insight commonly are abstract. They abstract from the irrelevant, the insignificant, the negligible, the incidental. They concentrate upon the relevant, the significant, the important, the essential.

But what is the relevant, the significant, the important, the essential? The answer depends immediately upon the insight, or set of insights, grounding the supposing, considering, thinking, defining, formulating. ~~More ult~~ Ultimately, one will say that the answer depends on which insight, or set of insights, is right. But we are not yet ready to tackle ultimate questions. Accordingly, we have to acknowledge, for the present, that the relevant and the irrelevant, the significant and the insignificant, the important and the negligible, the essential and the incidental, vary with one's insights. What at one time one thinks important, later, in the light of fuller insight, one will think unimportant. Inversely, what one used to think insignificant, now one may think significant; and what makes the difference is the advent of further ~~isn~~ insight.

Still, even for the present, this relative pronouncement is not the whole story. For if we restrict ourselves to the insights possible in mathematics, physics, chemistry, biology, sensitive psychology, and such sciences, then there are elements or components in sensible data and in images that always are regarded as irrelevant, insignificant, negligible, incidental. Such elements or components may be named the empirical residue. They are given as a matter of fact. But they are always disregarded when one concentrates on whatever one happens to think essential.

On four aspects of this empirical residue, something must now be said. They are 1) the individual, 2) the continuum, 3) place and time, and 4) the actual frequency of events.

5.1 Individuality Individuality pertains to the empirical residue. For whenever we understand anything, we would understand an exactly similar instance in exactly the same fashion. A different understanding would presuppose a difference in the data. It would presuppose the possibility of saying that the previous understanding would do, were it not for this aspects of the object. But, ex hypothesi, there is no aspect in which the second object differs from the first, and so there is no possibility of a different understanding. One may learn something new ~~about~~ when one turns to the second object; but one automatically learns it about the first object as well.

Thus, a first motor-car off the assembly line may be understood in terms of certain principles of construction and of operation. A second motor-car, similar in all respects, cannot but be understood in exactly the same fashion.

Nor is the issue changed essentially when one understands instances that are unique. In this case there is no possibility of apprehending a second object and understanding it in the same manner. But there is the possibility of apprehending the same object a second time; the data in the second apprehension will be similar to those of the first; ~~the~~ because the data are similar, the understanding has to be the same. The fact that the similar data are of the same object does not alter the underlying principle that our knowledge is so constituted that similar data have to result in similar insights with the consequence that, what is grasped by insight, is independent of the individuality of the data.

Thus, if the development of all life on this planet were comprehended in a single evolution, there would be no remainder of life on the planet to be understood in either the same or a different fashion. The object would be unique and unparalleled in our experience. None the less, the understanding would consist in grasping principles and laws in the combinations suitable for mastering the ~~data~~ enormous ranges of data, while knowledge of the ~~instance~~ unique instance would consist in observing the data to be understood.

~~In brief, nothing is explained by saying it is this instance. Inversely, whenever an explanation is grasped, that grasp is not by itself the knowledge of any instance. For the grasp is the viewpoint or insight that seizes upon the significant, relevant, important, essential; it finds this intelligibility in sensible presentations and imaginative representations; but it is no idle reproduction of what sense already senses or imagination imagines and so it is not a grasp not of the data but of an idea or form emergent in the data. Now it is a nice metaphysical question to settle whether such ideas or forms could be singular and individual. But the present issue is much simpler~~

Again, what is grasped by insight, may be named an idea or form emergent in sensible presentations or imaginative representations. But it is one thing to say that grasp of such an idea or form is knowledge of individuality, and quite another to say that within our experience there is found only one instance in which the idea or form can be grasped. If grasp of the idea or form were knowledge of individuality, then the individual would be known by understanding and it would not pertain to the empirical residue. But the mere fact that in some cases there is but a single, observable instance, in which the idea or form can be grasped, provides no evidence for the intrinsic intelligibility of individuality.

In ~~brief~~ brief, nothing is explained by saying that it is this instance. Inversely, in so far ~~we~~ as we grasp explanations, we know not instances but what may or may not be found in individual instances.

5.2 Continuity. The continuum pertains to the empirical residue.
Let us begin with a definition that departs from ordinary mathematical usage to suit our present purpose. A variable, x, will be said to be continuous in the range, $a < x < b$, if the values of x in ~~the~~ every part of the range form non-countable infinite sets. Next, a function, f(x), will be said to be continuous in a range if 1) x is continuous in the range and 2) for every distinct value of x there is a corresponding ~~distinct~~ value of the function. Finally, continuous functions possess a number of distinctive properties; hence, through the verification of these distinctive properties, it may be possible to verify the existence of continuous functions and so conclude to the existence of continua.

Now a continuum, in this defined and verifiable sense (which does not suppose a non-countable infinite set of observations) includes what cannot be counted because it cannot be ordered or systematized. By this inclusion of the non-systematic, a continuum clearly pertains to the empirical residue.

5.3 **Place and Time.** Place and time pertain to the empirical residue.

Space is a continuum of individual positions. Time is a continuum of individual instants. No position is any other. No instant is any other. And of both there are non-countable infinite sets. But the individual and the continuum both pertain to the empirical residue. So also, then, must place and time in their basic aspects.

Hence, when different experimenters, performing the same experiment at different places or times, obtain different results, then no one dreams of explaining the difference in the results by the difference in the place or by the difference in the time. The appeal always is, not to the place, but to something in the place, and not to the time, but to something at the time.

Indeed, if place or time made any difference, then each place and each time would have its own physics, chemistry, and biology. For if place were relevant, the laws in one place could not be the laws in another. If time was relevant, the laws at one time could not be the laws at another. Further, since places and times are non-countable sets, there would be non-countable sets of different physics, different chemistries, different biologies. Finally, none of these elements of these sets could be ascertained. For one cannot set up a whole physics, or a whole chemistry, or a whole biology, with the observations or experiments made at a point-instant.

~~It is to be noted, however, that one must not extend~~ However, it is only in their basic aspects that place and time pertain to the empirical residue. A place can be of singular importance, provided that importance rests not on a mere "there" but on a "something there." Such is the importance of the place occupied by a central mass in a gravitational field. Similarly, a time can be of singular importance, provided its importance rests not a mere "then" but on what happened then. Such is the importance of the initial moment in certain theories in the expanding universe.

5.4 **Actual Frequency.** Actual frequency pertains to the empirical residue.

The probability of tossing "heads" is 1/2. But in any series of actual tosses, one does not obtain a regular alternation of "heads" and "tails." Between the probability and the actual frequency, there is a divergence. Moreover, this divergence is random. It cannot be reduced to any law or mitigated by any reasonable expectation. It is non-systematic. It is to be known in each case only by actual observation. It too pertains to an empirical residue.

5.5 **The Significance of the Empirical Residue.** Let us now recall an initial restriction. The empirical residue was defined as always irrelevant from the viewpoint of insights in mathematics and the natural sciences. Why was this restriction imposed? Quite clearly, because in such a science as the theory of knowledge the notion of the empirical residue attains a systematic significance. For in a study of knowledge one attends systematically, not only to what is concentrated upon in abstraction, but also to what is regularly abstracted from. Theory of knowledge is a higher level science that takes as its materials the whole of the knowledge in other sciences.

Indeed, the theoretical account of the empirical residue is of considerable significance.

It is because insight abstracts from the individual that science is of the universal. It is because science is of the universal, that the observation of similarities is of such great heuristic importance.

It is because insight abstracts from the continuum by proceeding to the limit that the infinitesimal calculus is such a unique and powerful instrument in the construction of theories.

It is because insight abstracts from place and time that principles and laws are independent of place and time and that the expression of principles and laws ~~are~~ is invariant with respect to transformations of certain groups of coordinate systems.

It is because insight abstracts from the random divergence of the actual frequency that probability theory has its place among the instruments of scientific knowledge.

Generally, corresponding to each aspect of the empirical residue, there will be ~~xxxx~~ a remarkably powerful technique of intelligence in mastering the multiplicity of sensible data. Unfortunately, the discovery of the techniques has to be prior to the determination of the complementary aspect of the empirical residue. For while ~~the aspect of the empirical~~ all aspects of the empirical residue are given on the level of observation, still one can grasp them as pertaining to the empirical residue only by understanding the corresponding technique.