pp 21-23 replaced by pp 261-264

4.1 ~~4X3~~ The Elementary Paradox.

~~With the foregoing distinctions of size, length, and measurement, it seems possible to clarify the apparent paradoxes of the Special Theory of Relativity. We begin from a statement of relevant equations.~~

Let (x_1, t_1) and (x_2, t_2) be the coordinates ~~in~~ of a pair of point-instants, P and Q, in a reference frame, K.

Let (x'_1, t'_1) and (x'_2, t'_2) be the coordinates of the same pair of point-instants in a relatively moving frame, K', and let them from this viewpoint be named, P' and Q'.

On the Lorentz-Einstein transformation, writing

$$H = 1/(1 - u^2/c^2)^{1/2}$$

one relates the coordinates by the equations

$$x'_1 = H(x_1 - ut_1) \tag{1}$$

$$x'_2 = H(x_2 - ut_2) \tag{2}$$

$$t'_1 = H(t_1 - ux_1/c^2) \tag{3}$$

$$t'_2 = H(t_2 - ux_2/c^2) \tag{4}$$

Now consider two particular cases. So far, P and Q are any point-instants whatever; but in our first particular case we suppose that P and Q are the simultaneous positions of the ends of a standard measuring rod in the frame, K. Since the length of the rod is unity, and since the positions are simultaneous, we have

$$x_1 - x_2 = 1 \tag{5}$$

$$t_1 - t_2 = 0 \tag{6}$$

By subtracting equation (2) from (1) and equation (4) from (3) and substituting the values from equations (5) and (6), we have

$$x'_1 - x'_2 = H \tag{7}$$

$$t'_1 - t'_2 = -Hu/c^2 \tag{8}$$

so that, clearly, a unit length between simultaneous positions becomes on transformation a length that is not unity between positions that are not simultaneous.

In our second particular case we suppose that P and Q are the point-instants of successive seconds in a standard clock stationary relative to the frame K. Clearly,

$$x_1 - x_2 = 0 \tag{9}$$

$$t_1 - t_2 = 1 \tag{10}$$

whence, as before, by appealing to equations (1) to (4) and by substituting from (9) and (10), one obtains,

$$x'_1 - x'_2 = -Hu \qquad (11)$$

$$t'_1 - t'_2 = H \qquad (12)$$

so that a distance that is zero has been transformed into a distance that is not zero, and a time that is unity has been transformed into a time that is not unity.

Still, though distances and times are relative to reference frames, the four-dimensional interval is invariant. Let us name the interval, s, where

$$ds^2 = dx^2 - c^2dt^2 \qquad (13)$$

and in the present cases

$$s^2 = (x_1 - x_2)^2 - c^2(t_1 - t_2)^2 \qquad (14)$$

On substituting from equations (5) and (6), one finds that the interval of the rod in K according to the account in K is unity. Likewise, one substituting from equations (7) and (8), one finds that the interval of the rod in K according to the account in K' is unity. Again, on substituting from equations (9) and (10), one finds that the interval of the clock in K according to the account in K is **ic** [$i = \sqrt{-1}$]. Likewise on substituting from equations (11) and (12), one finds that the interval of the clock in K according to the account in K' is also **ic**.

Thus we have arrived both at the elementary paradox and at its solution. The elementary paradox arises from the contrast of equations (5) and (7) and again from the contrast of equations (10) and (12). The first contrast shows that the length of a rod in K is unity on the account in K ~~unity~~ but on the account in K' is greater than unity; and if K' finds a unit rod greater than unity, it seems to follow that his own rod is shorter. The second contrast shows that the length of a standard duration in K is unity in the account in K but is greater than unity in the account in K'; and if a unit of duration in ~~is~~ K is found to be greater than unity in K', it seems to follow that the unit in K' must be shorter.

However, if we began from rods and clocks in the system, K', we could establish the opposite conclusions with equal validity; for then it would seem to follow that the shorter units were in the system, K. Such is the elementary paradox.

~~The solution begins to appear when one notes that one can start~~ from any frame of reference, find that the four-dimensional interval of its rods is unity, find that the four-dimensional interval of its basic duration is **ic**, and on transformation find exactly the same values for these four-dimensional intervals in every other reference frame in uniform relative local motion.

The solution is that, in the context of Special Relativity, one may not regard rods as purely spatial and one may not regard clocks as purely temporal. The two ends of rods have positions at determinate times. The two ends of ~~durations occur at determinate places. It is true enough~~

What the paradox overlooks is the fact that, in the context of Special Relativity, one is not dealing with rods that are merely spatial or with clocks that are merely temporal. For, as has been seen, a standard rod determines an invariant four-dimensional interval of magnitude, unity; and a standard clock determines an invariant four-dimensional interval of magnitude, ic. Rods that determine an invariant four-dimensional interval must have a temporal component, and clocks that determine an invariant four-dimensional interval must have a spatial component.

Indeed, as appears from equations (5) and (6), in the reference frame, in which a rod lies between simultaneous point-instants, the invariant interval has a spatial component of magnitude, unity, and a temporal component of magnitude, zero. As appears from equations (7) and (8), in other relatively moving reference frames, the same rod determines the same four-dimensional interval, which, however, now has a spatial component of magnitude, H, and a temporal component of magnitude, - Hu/c^2. Concomitant with the variation of the spatial components, there is a variation of the temporal components. The rod in K by the account in K ~~ix~~ lies between simultaneous point-instants. The same rod in K by the account in K' lies between non-simultaneous point-instants. The spatial and temporal components, say [1, 0], transform to spatial and temporal components, [H, - Hu/c^2]. Inversely, the rod in K' by the account in K' will lie between simultaneous point-instants. But the same rod in K' by the account in K will lie between non-simultaneous point-instants. In this case, spatial and temporal components, [1, 0], transform to spatial and temporal components, [H, Hu/c^2], for the sign of the relative velocity, u, changes.

~~Again, in the reference frame~~

Again, as appears from equations (9) and (10), in the reference frame, in which the beginning and the end of a standard duration occur in relatively the same position, the invariant interval of magnitude, ic, has a spatial component of magnitude, zero, and a temporal component of magnitude, unity. As appears from equations (11) and (12), in other relatively moving frames of reference, the same duration determines the same invariant interval, which, however, now has a spatial component of magnitude, - Hu, and a temporal component of magnitude, H. Again, there is concomitant variation of spatial and temporal components. A standard duration in K by the account in K has components [0, 1]; the same duration in K by the account in K' has components [- Hu, H]. Inversely, a standard duration in K' by the account in K' will have components [0, 1]; but this duration in K' by the account in K will have components [Hu, H].

The elementary paradox results from a cumulation of oversights. It disregards the invariant interval fixed by any rod for all reference frames and the invariant interval fixed by any clock for all reference frames. It disregards four accounts of two rods to consider only two rods, and it disregards four accounts of two clocks to consider only two clocks. Finally, it disregards the temporal component that pertains to rods and the spatial component that pertains to clocks.

Still, if the elementary paradox is to be set aside as a gross over-simplification, there remains in its entirety the problem of working out a coherent account of the notion of measurement compatible with the complexity of Special Relativity. To this task we must now address our attention.