These leaves* seem to have been removed from the 1953 typescript after summer of 1954 and BL's conversations w. M. Longman (see letters of summer + fall 1954 - also visit here by Eric O'C. dated Nov. 7/54).

— F.E.C., June 12/72.

* P. McS., Oct. 7th/72, lists these as 2b to 2f.

This is what is to be deleted from Chapter X #8
Reflective Understanding - Mathematical Judgments -
Replace by what follows i.e. pp52-58 on original typing.

pp52-59 on carbon copy

others and so relating and defining them. Further, we may presuppose each department of mathematics to be formalized, that is, to be stated as a set of definitions, postulates, and deductions. Finally, we shall presuppose that there are other formalizations, equally rigorous, ~~similar to the others~~ equally elegant, but in fact not members of the mathematical series. Our problem thus becomes the question, How is one to recognize some formalizations as mathematical and others as not mathematical?

Two answers are offered. The first will be named psychological, for it is cast in terms of insights and reflection. The second will be named conceptual, for it attempts to lay down rules that characterize the mathematical series.

The psychological answer runs as follows.

In any mathematical department, terms are related by operations, and operations are governed by rules; but the rules are the expression of clusters of insights, and the clusters of insights stand in a psychological series. There is a laborious process named "learning mathematics". It consists in gradually acquiring the insights that are necessary to understand mathematical problems, to follow mathematical arguments, to work out mathematical solutions. That acquisition of insights involves a succession of higher viewpoints. But each higher viewpoint is related to previous, lower viewpoints. As was argued in a previous section, the symbolic representation of operations in the lower field provides the images in which intelligence grasps the new idea of the set of rules governing operations in the higher field. Hence, though the successive departments of the mathematical series are discontinuous from a logical standpoint, for they suppose different definitions and postulates, still they are continuous from a psychological viewpoint, for one gets the idea of the later in working at the earlier. Such psychological continuity defines the mathematical series. It settles which formalizations are mathematical and which are not.

However, on this psychological solution, only the competent mathematician can judge. Just as common sense judgments are the province of men of common sense, just as the probability of an empirical theory can be estimated only by the man familiar with that branch of inquiry, just as one relies on men of experience, on experts, on specialists in their respective fields, so in determining what is mathematics one has to appeal to the mathematician. The grounds for this position are quite general. Other judgments depend on judgments that settle which insights are correct. But which insights are correct can be settled only by the familiarity and mastery that stands as a limit to the self-correcting process in which previous insights give rise to further questions, and further questions give rise to complementing insights. As in other fields, so too in mathematics, it is the man who has been through the self-correcting process of learning that possesses the familiarity and intellectual mastery on which judgment has to rest.

The conceptual answer is the conceptualization of the psychological answer. We recall that analytic principles are analytic propositions whose terms are existential: and that existential terms are terms that occur in factual judgments. Further, serial analytic principles are analytic propositions whose terms are serially existential; and terms are serially existential in a series of manners.

What is that series of manners? As the reader will suspect, it is the series that corresponds to the series of mathematical departments; again, it is the series that corresponds to the psychological series of learning mathematics. But our present task is to attempt to assign rules by which that series can be recognized, and our procedure will be, first, to offer illustrations of the serially existential and, secondly, to assign rules that characterize the series of manners in which terms may be serially existential.

A first illustration of the serially existential is offered by the positive integers.

Obviously, not all positive integers are existential, for not all occur in factual judgments. Indeed, no matter how many millions of them do so, there is an infinity of other positive integers that do not.

On the other hand, the whole series of positive integers is grounded in the existential. For in the first place, many positive integers do occur in factual judgments; and, in the second place, the generative principle of the series, the operation of adding "one more", also occurs in factual judgments.

The first case of the serially existential is, then, a series of terms that satisfies two conditions. Some of the terms are existential. The generative principle of the series is in some instances existential.

A second illustration of the serially existential is offered by the totality of "real" numbers.

First, some of the real numbers, namely, the positive integers, are serially existential. Secondly, from the positive integers by appropriate operations, one can reach any of the real numbers. Thirdly, these appropriate operations are in some instances equivalent to existential operations.

The third point may need elucidation. "Adding" in the sense in which it certainly occurs in factual judgments can base definitions of substracting, multiplying, dividing, powers, and roots. However, these operations, so defined, will not yield the totality of the real numbers. What is needed is the algebraic generalization of these operations, and one may doubt that such generalized operations occur in factual judgments. But what is beyond doubt is that in certain instances both the initial and the generalized operations yield the same results. The square root of 16 is 4, whether you define square root by a theory of indices, or take it as the inverse of "multiplying a number by itself" and "multiplying a number by itself" as "adding that number the same number of times and adding the resultant aggregate".

A third illustration of the serially existential is offered by geometry. Let us say that Euclid's geometry consists of elements and constructions. Now such elements as points, finite straight lines, finite planes, in their defined sense, do not occur in factual judgments. Still what approximates to them does occur in factual judgments. Again, the exact execution of Euclidean constructions, if it occurs, cannot be ascertained as in fact, exact.

Still what approximates to such exact execution does occur in factual judgments. In this case, the serially existential will be a totality of terms, some of which approximate to the existential, and the rest of which can be generated by constructions that approximate to the existential. From that basis one can proceed to add further members of the serially existential. ~~Thus, lines produced to infinity, planes indefinitely extended.~~ Thus, lines or planes extended to infinity will be serially existential because their starting-point and their generative principle are serially existential. Again, geometries that include the Euclidean as a particular case would be serially existential because of that case. On the other hand, specifically non-Euclidean geometries could be serially existential on the ground that, like the Euclidean, they approximate to the existential. Finally, one might conceive a grand series of geometries and hold all to be serially existential because some are serially existential and because the generative principle of the lot is serially existential inasmuch as it relates the few that are serially existential.

Three illustrations of the serially existential have been offered. It is time to try to grasp the notion itself.

First, then, the serially existential takes the existential as its starting-point.

Secondly, the existential is merely a starting-point. It represents the knowledge of present factual judgments. But present factual judgments are limited by present ignorance. The vast field of the unknown that remains to be known is worth exploring. If, so to speak, the mathematician has one foot solidly placed on the earth, the other is raised and groping for further foot-holds.

Thirdly, this exploration is conducted systematically. Its rule is to be perfectly general. Some positive integers occur in factual judgments. Others might occur. Which ones might, we do not know. Explore the lot. In like manner, the totality of real numbers is envisaged, the totality of geometries, and so forth.

Fourthly, the sets of operations with their terms, that are explored with perfect generality, are referred to the known existential, as the complete series to the incomplete, as the general to the particular, as the ideal to the approximate, moreover, the reference may be direct or mediated, and the mediation may involve any number of steps.

Fifthly, the series of different manners of the serially existential is not closed but open. Future developments of mathematics are still possible. Their precise nature cannot be foreseen, and so their precise reference to the existential is to be left to future discovery or invention.

Sixthly, if every department of mathematics is correctly referred to the existential, then it will always remain a department of mathematics. Still, this does not preclude revision of the manner in which that reference to the existential is conceived; for subsequent developments may involve more satisfactory manners of conceiving earlier developments.

Seventhly, since sound judgment presupposes an accumulation of insights, sound judgment on the validity of new departments of mathematics is to be expected, not at their inception when the comprehensively the existing universe. Finally, correct mathematics can have their exemplar in the divine mind not only as correct conceptions of possible things but also as correct human operations to envisage with complete generality the series of terms some of which in some fashion present an aspect that is existential.

Second, what about axiomatic coherence?

The real ground of coherence of a set of axioms is the cluster of insights in an intelligence. Inasmuch as a department of mathematics emerges through the development of mathematical understanding, its coherence seems to be guaranteed automatically. On the other hand, a merely formalizing approach to mathematical foundations has the problem of coherence on its hands.

Thirdly, what about the principle of excluded middle?

Our requirement of reference to the existential is opposed in principle to setting up objects that are defined specifically by the mere negation of known mathematical properties. For such negation does not imply that there is some reference, however tortuous, to the existential. However, this opposition in principle does not deny 1) that such objects can be explored logically or 2) that such logical exploration may be a useful complement to mathematical thought. Opposition in principle is limited to the contention that such appendages are mere appendages, that mathematics in the full sense is something distinct.