Appendix to Section 9. On the assumption that all units of enterprise begin turnover 1 simultaneously and end turnover n simultaneously, it is possible to construct a simple mathematical model of circuit acceleration. One may write

$$R' = \sum_{i'} \sum_{1}^{n} r_{ij}$$

$$O' = \sum_{i'} \sum_{1}^{n} o_{ij}$$

$$S' = \sum_{i'} \sum_{1}^{n} s_{ij}$$

where the unit of enterprise, i, in turnover, j, increases its monetary circulating capital by $s_{ij}$, makes initial payments, $o_{ij}$, and receives final payments, $r_{ij}$, and each double summation is taken in each unit of enterprise from turnover 1 to turnover n and then with respect to all basic units of enterprise, i'.

Now $r_{ij}$ is zero in all units of enterprise that do not deal immediately with final buyers. Let $T_{ij}$ be transitional payments received and $t_{ij}$ be transitional payments made by the unit of enterprise, i, in turnover j. Further, let j' denote the turnover immediately preceding turnover, j. Then, since the increment in monetary circulating capital may be equated to the excess of payments made in turnover, j, over payments received in turnover, j', one has

$$r_{ij'} + T_{ij'} = o_{ij} + t_{ij} - s_{ij}$$

Submitting this equation to a double summation, one has

$$\sum_{i'} \sum_{0}^{n-1} (r_{ij} + T_{ij}) = \sum_{i'} \sum_{1}^{n} (o_{ij} + t_{ij} - s_{ij})$$

where the difference between turnovers j' and j is covered by the difference of the limits. However the limit on the left-hand side may be assimilated to that on the right-hand side by introducing dR' and dT' defined by the equations,

$$dR' = \sum_{i'} (f_{in} - f_{i0}) \qquad \left[\text{so that } \sum \sum_{1}^{n} f_{ij} - \sum \sum_{0}^{n-1} f_{ij} = dR'\right]$$

$$dT' = \sum_{i'} (T_{in} - T_{i0})$$

where, since turnover 0 is the last turnover of the previous interval, dR' is the difference in the turnover magnitude of basic final payments at the beginning and at the end of the interval; and dT' is the similar difference in the turnover magnitude of basic transitional payments. Further, since in the aggregate transitional payments made are identical with transitional payments received, one has

$$0 = \sum_{i'} \sum_{1}^{n} (T_{ij} - t_{ij})$$

so that all summations may be eliminated and one may write

$$DS' = dT' + (DO' - DR') + dR' \qquad (15)$$

and by changing (') to (") one has a parallel equation for the surplus circuit.

With DG at zero and DD' at zero, (DO' - DR') can be no more than a lag and, as will appear later, this lag unfortunately tends to be zero. Again, dT' and dR' will be of the same sign: final payments are not increasing when transitional payments are decreasing nor vice versa. Thus, excess transfers to or from supply, DS', tend to equal the sum of the increments of aggregate turnover magnitudes in final payments and in transitional payments. Of these two, the increment in transitional payments will be the larger, since for each sale at the final market there commonly is a sale at a number of transitional markets.

In the summations, turnover magnitudes appear directly but turnover frequencies only indirectly, inasmuch as the number of turnovers, n, per interval increases in the case of any given unit of enterprise. An increase of the number of turnovers per interval would have a great effect on DR' and DO', since it increases the number of instances of $f_{ij}$ and $o_{ij}$ to be summated. But it need have no effect on DS', since $s_{ij}$ may be positive or negative and so cancel in the aggregate of instances. Finally, increasing turnover frequency, of itself, has no effect on dR' or dT', since these terms represent the difference between two turnover magnitudes, and increasing frequency only puts further apart the two magnitudes compared.

Thus, in pure theory DO' and DR' might accelerate to any extent while DS', dR', dT' each remained at zero. It would be a pure frequency acceleration. The contention of the preceding pages was that such pure frequency acceleration has conditions that are difficult to realize, and that the history of the development of money points to a preponderant role of increasing turnover magnitude in circuit accelerations.

10. The Theoretical Possibility of Measurement of the Productive Process. In the three preceding sections classes and rates of payment were defined to make possible a statement of conditions of acceleration of the monetary circuits. Evidently it is desirable to complete this list of conditions by bringing in a consideration of the underlying acceleration of the productive process itself. But before this can be done, at least a method of defining the measurement of such acceleration must be provided. It is not necessary that any actual measurement be undertaken, or even that a method which statisticians would find practicable be assigned. But it is necessary that we have a clear and definite idea of what we are discussing when we speak of an acceleration of the productive process.

The problem of the present section may be put as follows. Consider two successive and equal intervals of time, long enough to be representative, yet not so long that much averaging is required. Suppose that in the first of the two intervals, objects of a class, i, were sold in the quantity, $q_i$, and at an average price, $p_i$ + $dq_i$, and at the average price, $p_i + dp_i$. Suppose, further, that there are n such classes, that the aggregate payment for them in the first interval was DZ and in the second interval was $DZ + D^2Z$, so that

$$DZ = \sum p_i q_i \tag{16}$$

$$DZ + D^2Z = \sum (p_i q_i + p_i dq_i + q_i dp_i + dp_i dq_i) \tag{17}$$

where all summations are taken with respect to all instances of i from 1 to n. The question is, How much does the increment in the rate of payment, $D^2Z$, result from price increments, $dp_i$, and how much does it result from quantity increments, $dq_i$? In other words, can one define two numbers, say

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", and in the second interval objects in the same class were sold ~~at an ave~~ in the quantity, $q_i$"

P and Q, such that P varies with a set of numbers, $p_1$, $p_2$, $p_3$,.... and Q varies with another set of numbers, $q_1$, $q_2$, $q_3$,....

An universally valid answer to this question may be had when P and Q are not mere numbers but vectors in an n-dimensional manifold. Let P and Q be the vectors from the origin to the points ($p_1$, $p_2$, $p_3$,...) and ($q_1$, $q_2$, $q_3$,...) respectively. Then any variation in the price pattern, that is, in any ratio of the type, $p_i/p_j$, will appear as a variation in the angle between the projection of P on the plane "ij" and the axis "j". Similarly, any variation in the quantity pattern will appear as a parallel variation in an angle made by a projection of Q. But besides such variation in price pattern or in quantity pattern there may be general increases or decreases in prices or in quantities. The latter appear as positive or negative increments in the absolute magnitudes of the vectors, for

$$P^2 = \sum p_i^2 \tag{18}$$

$$Q^2 = \sum q_i^2 \tag{19}$$

that is, the length of the vector, P, is the square root of the sum of all prices squared, and the length of the vector, Q, is the square root of the sum of all quantities squared. Thus, one may suppose two n-dimensional spheres of radii P and Q respectively. The vector from the origin to any point in the first "quadrant" of the surface of such spheres represents a determinate price pattern or quantity pattern. On the other hand, variation in P and Q is variation in the size of the spheres.

Now there is a well-known theorem, called the "dot product", which enables us to equate DZ with P and Q, whence

$$DZ = \sum p_i q_i = PQ \cos A \tag{20}$$

where A is the angle between the vectors, P and Q. Thus, variation in DZ depends not only on the magnitude of P and Q but also on the price and quantity patterns as represented by the angle A between the vectors. This is evident enough, since it makes a notable difference in DZ whether large or small instances of $p_i$ combine with large or small instances of $q_i$, and such combination is ruled by the relative price and quantity patterns, to appear ultimately in the angle A.

Next, consider the second interval in which the vector, P, increases to (P + dP), the vector Q to (Q + dQ), and the angle A to (A + dA). Then

$$(P + dP)^2 = \sum (p_i + dp_i)^2 \tag{21}$$

$$(Q + dQ)^2 = \sum (q_i + dq_i)^2 \tag{22}$$

$$DZ + D^2Z = (P + dP)(Q + dQ)\cos(A + dA) \tag{23}$$

From equations (20) and (23) one obtains for $D^2Z$ the expression

$$D^2Z = PQ\big[(dP/P + dQ/Q + dPdQ/PQ)\cos(A + dA) - (\sin A \sin dA/2)/(dA/2)\big] \tag{24}$$

Thus, $D^2Z$ depends not only on the initial quantities, P, Q, A, and the increments in absolute magnitude, dP and dQ, but also upon changes in the relative price and quantity patterns as represented by the angle, dA.

From equations (18) to (20) it can be shewn that if all prices and all quantities change in the same proportion, then there is no change in cosA, so that the angle, dA, is zero. Further, dA is again zero whenever there is compensation for deviation of change from the same proportion, inasmuch as some prices or quantities change more and others less than a strictly proportionate change would require. But whenever dA is zero or very small one may write,

$$D^2Z = PQ\big[(dP/P + dQ/Q + dPdQ/PQ)\cos A - \sin A\big] \tag{25}$$

so that the increment, $D^2Z$, then depends solely on the increments dP and dQ and the initial quantities, P, Q, A.

Now the significance of the foregoing is purely theoretical. The question has been about the possibility of price and quantity indices. The only relevant common measure of tons of iron ore, ton-miles of transportation, kilowatt hours, and so on, lies in their prices; but prices themselves are subject to change; hence if it is possible to measure the acceleration of the productive process, it has to be possible to differentiate between price variation, price pattern and quantity pattern variation, and the pure quantity variation of the productive process. The foregoing discussion has aimed at showing that, without ever lapsing into meaninglessness, it is always possible to make such distinctions.

However, the definitions that have been given are rather elaborate. When change is gradual, it will be sufficient to use the following approximate definitions of P, Q, dP, and dQ, namely,

$$PQ = \sum p_i q_i \tag{26}$$

$$PdQ = \sum p_i dq_i \tag{27}$$

$$QdP = \sum q_i dp_i \tag{28}$$

$$dPdQ = \sum dp_i dq_i \tag{29}$$

so that

$$D^2Z = PQ\,[dP/P + dQ/Q + dPdQ/PQ] \tag{30}$$

as results by referring back to equations (16) and (17) which pertain to the statement of the problem. On this definition one obtains different values for P and Q and they may be termed "weighted averages" as opposed to the previous "vectorial averages". The difference is most apparent in the respective equations for $D^2Z$: equations (24) and (25) contain all the relations of equation (30) but add to the latter a qualification by introducing a trigonometric function of the angle A.

The greater simplicity of the weighted averages is not without its drawbacks. Equations (26) to (29) have to be taken simultane[redacted] must be consistent; it follows that as the four left-hand side expressions are in a fourfold proportion [PQ / PdQ = QdP / dQdP] so also, for consistency, the four right-hand side summations must be in a fourfold proportion. This condition obviously restricts the validity of the definition by weighted averages: in the rare cases when the summations are proportionate, the definition is exact; when the summations are approximately proportionate, the definition is no more than an approximation; when the summations are not even approximately proportionate, the definition involves a contradiction and so is meaningless.

Naturally, a theorist is ill at ease when dealing with objects whose definition can lapse into meaninglessness and usually is at best approximate, especially when there is no saying what it approximates to. On that account one may well prefer to regard equation (30) is an alternative expression for equation (25): both of these equations have parallel variables, $D^2Z$, dP, dQ, though the latter adds a further initial quantity, A, to P and Q; apart from the additional initial quantity of (25), both relate variables and initial quantities in the same way; both are in the general case approximations; and both have parallel conditions of approximation, namely, a fourfold proportion involving sets of prices, quantities, price increments, and quantity increments. This parallelism should seem sufficient to provide an answer to the embarrassing question, To what do the weighted averages approximate? One may say that they are a simplified approach to the conceptually exact vectorial averages. So much, then, for the theoretical problem of the measurement of the acceleration of the productive process: from rates of payment, DZ, and their increments, $D^2Z$, it is possible to proceed to rates of production, Q, and their increments, dQ.

There remains the question of the application of this method of measurement to the basic and surplus stages of the productive process. In general the discussion will centre on hypothetical smooth trends of expansion, so that instances of $dp_i$ and $dq_i$ will all be relatively small and the definition in terms of weighted averages will be available. The main indices to be employed will be, P', the basic selling price index and dP' its increment, Q', the index of basic quantities sold in the given interval and dQ' its increment, P", the surplus selling price index and dP" its increment, and Q", the index of surplus quantities sold and dQ" its increment. These indices are calculated from rates of payment at the basic and surplus final markets, as follows:

$$DE' = P'Q' \qquad (31)$$

$$D^2E' = P'Q' \left[dP'/P' + dQ'/Q' + dP'dQ'/P'Q'\right] \qquad (32)$$

$$DE'' = P''Q'' \qquad (33)$$

$$D^2E'' = P''Q'' \left[dP''/P'' + dQ''/Q'' + dP''dQ''/P''Q''\right] \qquad (34)$$

At times of great and abrupt change, when weighted averages cease to have a meaning, the meaning of the indices may be salvaged by shifting to the definition in terms of vectorial averages and adding to equations (31) to (34) the appropriate trigonometric functions of A and dA. On the other hand, in discussing equations of the type of (32) and (34), one may ignore the third quotient on the right-hand side, often because it is relatively small, always because it is merely the product of the first two quotients and so does not add a further factor of variation that is different in kind.

Since Q' and Q" refer to quantities sold at the final markets, they have to be corrected by acceleration coefficients, a' and a", to give quantities under production during the contemporaneous interval. Thus, when basic quantities sold are Q', basic quantities under production will be a'Q'; similarly, when surplus quantities sold are Q", surplus quantities under production will be a"Q". Estimates of the acceleration coefficients proceed in two steps. First, one considers the series of indices for final sales over a number of intervals, say, $Q'_1$, $Q'_2$, $Q'_3$,... If these are about equal, the acceleration coefficient will be unity; if they are an increasing series, then $a'_1$ will be greater than unity; if they are a decreasing series, then $a'_1$ will be less than unity. Second, one adverts to the influence of speculative anticipations: the current rate of production is not based on actual but on anticipated future rates of final sales; further, when prices are rising, there is an advantage in buying long in advance, and when prices are falling, the advantage lies with minimum inventories; finally, there is a cumulative effect whenever there is a series of transitional markets, for each successive market tends to count the speculative increments of demand of later markets as part of the objective evidence, to add on a further speculative increment, and to pass on a cumulatively inflated or deflated demand to earlier markets. Hence one may characterize the acceleration coefficients as greater or less than unity according as the stages of the process are accelerating or decelerating, as notably greater than unity when current production is expanding speculatively, and perhaps as tending to be notably less than unity in the liquidation of a crisis.