Bibliography JLS Analytic Lonergan

II

# The Development of Mathematical Logic

1. The development of ML has been the pursuit of an ideal, viz., a rigorous hypothetico-deductive system that from minimal suppositions would embrace the whole of known mathematics.

2. The ideal has been formulated as an axiomatic system or logical formalization.

It distinguishes terms and propositions, divides both into derived and not-derived, conceives not-derived as relative to system, and names them primitive.

Derived terms are defined by primitive.

Derived propositions are deduced from primitive.

Rules of derivation must be stated explicitly, and no derivations are admitted except in accord with stated rules.

Let P and Q denote two LF's, and let p be any proposition that can be constructed in P.

Then P and Q are equivalent, if primitive of P are derived in Q and primitive of Q are derived in P.

P is complete, if one can derive either p or Np.

P is coherent, if one cannot derive both p and Np.

The primitive propositions of P are independent if no one can be derived from the others.

The primitive propositions of P are elegant if they offer the simplest basis for deriving in the simplest manner all the propositions of P.

3. Principal lines of endeavor.

a Axiomatic set theory: Zermelo - Fraenkel - von Neumann

b Whitehead-Russell, Principia mathematica; aims to base whole of mathematics on logical axioms; a magnificent unitary view that remains one of the principal directions.

However in both first and second editions there is a non-logical axiom of infinity.

In first addition there is also a "theory of types" (to avoid paradox of class of classes that do not contain themselves) and an axiom of reducibility (to make possible Dedekind's definition of real number, excluded by theory of types).

In second edition the axiom of reducibility is eliminated and there is employed a weakened theory of types that eliminates syntactical but not semantical paradoxes.

c Hilbert proposed a two-level approach.

First, a formalized deduction of the whole of mathematics from mathematical axioms; on this level there were to be admitted infinities of objects and of operations.

Secondly, a metamathematics that on a strictly finite basis would investigate logical properties (especially consistency) of the first mathematical level.

Results: short term, geometry worked out with axioms verified intuitively in model that supposes validity of counting numbers; arithmetic could be shown to be consistent only if some axioms were omitted or all weakened. However, as will appear, this has proved most fruitful line of inquiry.

d Intuitionistic school: Brouwer, Heyting

Insists that LF is only tool, that mathematics is essentially constructive, that excluded middle cannot be invoked indiscriminatly. Program involves lopping off more of classical mathematics than mathematicians are ready to sacrifice.

e Gonseth; review Dialectica

Tends to conceive axiomatic ideal just an outdated Euclidean avatar; insists on development, interaction between maths and cultural movements; relativist in tone.

f Bourbaki group: Hilbert's first level; metamathematics is a separate department of no particular interest to mathematician; weak point that rigid axiomatic structure neither accounts for past development of maths nor opens way to developments of future.

4. Godelian limitations.

Jean Ladriere, Les limitations internes des formalismes, Louvain (Nauwelaerts) and Paris (Gauthier-Villars) 1957. Pp. 702

There have been demonstrated a series of theorems setting limitations to the possibility of reaching the ideal of the rigorously deductive mathematical system. The general form of the argument in such cases is approximately as follows:

a An LF is a symbolic technique capable of representing a manifold of deductive sequences.

Consider an LFL and an LFM, which symbolically are identical or sufficiently parallel, but differ inasmuch as LFL is interpreted logically while LFM is interpreted mathematically.

b. Now in mathematics there exist non-enumerable sets, i.e., aggregates that do not admit a one-to-one correspondence with the positive integers, and so cannot be enumerated (counted).

Hence, to suppose that such a set is enumerable (e.g. the set of infinite decimals) results in a contradiction, and this contradiction can be demonstrated.

c With sufficent ingenuity it is possible to make the LFL sufficiently parallel to the LFM so that the proof of non-enumerability in LFM is is matched by a proof of logical impossibility in LFL.

In other words, the proof that the proposition "K" has been enumerated" is contradictory is paralleled by a proof that the proposition "K" is a theorem, or "K" is a soluble problem, or "K" is true, or "K is definable" is contradictory.

d Such theorems are extremely complex; they have been worked out in a variety of manners; they arise when the LFL is sufficiently powerful to represent the theory of division, resolution into prime factors, and the unicity of such resolution.

e Their proximate significance was the refutation of Hilbert's proposal to settle the logical validity of arithmetic on a finitist basis.

Godel's demonstration was followed by a demonstration by Gentzen that arithmetic was non-contradictory, where however the LFL had to employ transfinite induction.

f The ultimate significance, however, of such Godelian limitations seems to be the same as of inverse insight; cf. irrationals, transcendental numbers, Galois on fifth degree equations, Newtoh's first law.

5. The Transcendence of Godelian Limitations.

a Mere avoidance: J. S. Myhill (JSL 15(1950) 185-196) avoids such consequences by employing a logic without quantification and without negation.

b Use of indefinitely large stratifications (analogy)

Church: "implication" and "quantification" take on different meanings on different strata

Curry: similar procedure re his basic notion of canonicity.

c Skolem paradox shows that by different modes of stating one-to-one correspondence, "enumerable" takes on different meanings.

d L. Henkin's study of relations between LFL and models showing that LFL lacks absolutely definite meaning.

e Hao Wang: indefinite series of sub-systems; at each level new resources of construction and new meaning for enumerable; the consistency and the theory of truth for any level, m, demonstrable at level (m + 2).

f Significance: ideal of ML moving from static and closed to analogous and opeh.