The Problem of Exactitude in Geometry[1]

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Part I[2]3

Two articles that appeared in previous numbers of this journal dealt with 'the scholastic philosophy of geometrical knowledge'[3] and 'the problem of necessity in geometry.'[4] The first of these, which will be cited here as 'the introduction,' set out in general terms the problems of this area of cognitional theory and stressed the need to undertake in inquiry into it. The second article attempted to solve one of these problems, that of the necessity belonging to geometrical knowledge, as to both the facts and their justification.

In the present article we shall deal with another problem needing to be solved, namely the problem of exactitude, in which geometry differs from the outset from arithmetic, where exactitude does not constitute an enigma. Not everything that is involved in this problem requires a solution by the same means, as our exposition will show. For there are certain things whose solution requires a consideration of the extended as the 'principle of indivituation', and certain others that do not depend on this means. This article will address these latter points; at a later time, God willing, we will treat other questions, those that depend on an examination of the principle of individuation.

§ 1. Statement of the Problem

1. *The nub of the problem.*

We recall from the introduction that the problem of exactitude consists in the following. Those elements that geometry claims as its proper objects – points, lines, surfaces and their mutual relations, the form of its figures, e.g., the straightness of its lines, the equality or proportion of its figures – all these are held to be absolutely exact. The data of the senses, however, from which classical geometry deduces its origin, do not possess this exactitude. Neither sense nor imagination can perceive points and lines but only the extended body with its three (or at least two) dimensions. Nor can they distinguish lines that are slightly curved from those that are straight; as a result, they can never determine that a given line is actually straight. They will be able *roughly* to perceive the equality between two lines or surfaces, but not *exactly*; and the same is to be said for proportions. They cannot see things that are far distant, and yet geometry deals with what extends into an infinite distance.

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[1] "De problema exactitudinis geometricae," *Gregorianum* 20 (1939) 321-350.
[2] There is no *Pars II* in the text as printed. (Tr.)9
[3] "De philosophia scholasticae cognitionis geometricae," *Gregorianum* 19 (1938) 498-514.
[4] "De problemate necessitatis geometricae," G*regorianum* 20 (1939) 19-54. These articles were preceded by "De origine primorum principiorum scientiae," *Gregorianum* 14 (1933) 153-184.

If, therefore, geometry in its origin depends on sense data, how can this transition to perfect exactitude be accounted for? Is it, perhaps, an illusion?

Note well: regarding the problem of necessity, the question was especially, though not only, about the *nexuses* between notions, which were affirmed in judgments; here we deal with both these questions, not with *nexuses* only, but no less with the existence of the *notions* themselves.

2. *Some historical points.*

Philosophers do not usually pay much attention to this problem, and many of them have perhaps totally ignored it; but for that reason modern mathematicians have done so all the more.

A. *Philosophers.* Aristotle was well aware of the problem but said very little about it. In the *Posterior Analytics* I, 31, he touches on the matter where he says, "Even if it were possible to think of a triangle having two right angles, we would surely ask for proof and would not, as some contend, have knowledge" [*scientiam*].[5]

The main point to which we must now attend is not the consequent of this conditional, namely, that for such *knowledge* a demonstration would be required, because of the necessity and universality of this proposition concerning a triangle; but rather let us attend to this: Aristotle quite rightly points out the difference between the data of the senses, which never exactly indicate the sum of the angles, and geometrical truth, which is clear to the intellect. And St Thomas in his commentary (lect. 42, n. 7) also has an excellent comment on this when he adds, "This is an example of what cannot be perceived by the senses." For perfect equality between the sum of the angles of a triangle and two right angles can in no way be perceived by the senses nor arrived at by any method.

This proposition, therefore, about the sum of the angles of a triangle can never be the *first* intuitive principle; in this case *direct* transition from the data of the senses to an exact affirmation of the intellect can never the justified, not only because we do not here directly intuit the nexus that is necessary for all triangles but also because exactitude on the part of the judgement would not be justified here. If a transition to exactitude is possible, it will have to be found from other data; and the problem is: are there such data?

In antiquity Proclus frequently stressed the inexactness of sense data; and the reason for this is that in this matter he rejects Aristotle's theory of abstraction and adheres rather to Plato's position. For he supposes, first, that Aristotle's theory cannot account for the apodictic certitude of geometry (*On Euclid's Elements*, ed. Friedlein 12, 14; 4k9, 17 sqq., 140, 4); in this he is wrong, as we saw in the first chapter. But above all, he holds that the problem of exactitude

[5] 87b 35.

cannot be solved in this way (ibid. 12, 14-22; 49, 12-17; 139, 26-104, 7, etc.). Hence, although he admits the theory of intelligible matter, he gives it a Platonic explatation.[6]

The problem of exactitude has received little attention among modern philosophers. For the theories of Berkeley and Hume, who did not clearly distinguish intellectual knowledge from sense knowledge, see our *Cosmologia*, p. 36.[7] Kant, as was mentioned in the introduction (p. 505), seems to have no distinct awareness of our problem.

For Stuart Mill, who followed the theories of Berkeley and Hume, the problem did not exist, because the exactitude that geometry claims to attain does not exist, either in nature or in the mind. Or perhaps one should say that Mill, because he holds that the problem of the transition from inexactness to exactitude is insoluble, therefore denies that exactitude exists either in nature or in the mind. He says: "There exist no points without magnitude, no lines without breadth or that are perfectly straight, no circles with all their radii exactly equal, or squares with all their angles perfectly right.' Moreover, suich figures are impossible. '... according to any test we have of possibility, they are not even possible.' They are not possible in nature because of the constitution of our planet at least, if not of the universe.' Nor are they possible in the mind, since there we have only images of what we know through experiece. 'The points, lines, circles and squares which any one has in his mind, are (I apprehend) simply copies of the points, lines, circles and squares which he has known in his experience. Our idea of a point I apprehend to be simply our idea of the *minimum visibile*, the smallest portion of surface which we can see. A line, as defined by geometers, is wholly inconceivable."[8]

And he concludes: "The peculiar accuracy, supposed to be characteristic of the first principles of geometry, thus appears to be fictitious."[9]

For Stuart Mill the problem of exactitude does not exist, because there is no such thing as exactitude itself, either in the real world of nature or in the possible world or in the human mind. It is clear that this position is false from the outset; for the human mind for more than twenty centuries has perfectly distinguished the exact concepts it has of these figures from images, and has reasoned intelligibly about those exact figures. But we will touch on these various points later, among which there is one of greater importance: Mill has confused different kinds of exactitude.

B. *Mathematicians*. In their critique of modern mathematics, mathematicians seldom spend much time on the problem of necessity; they focus all their attention on examining the question of exactitude and in their examination seem to arrive at this conclusion, that the transition from the inexact sense data of geometry to exact classical geometry is illegitimate.

[6] Cf. Cl. Baeumker, *Das Problem der Materie in der Griechischen Philosophie*, pp. 422 ff.
[7] Petrus Hoenen, S.I., *Cosmologia*. Rome: Gregorian University Press, 3rd ed., 1945.
[8] J. Stuart Mill, *System of Logic* I, ed. 5 (1862) p. 255.
[9] Ibid. p. 257.

Therefore they try to construct an analysis in a purely arithmetical manner. Whether, once this analysis has been constructed, its *application*, equivalent to classical geometry, is possible, will be a further problem.

They do not usually consider the problem of necessity. They admit necessity in arithmetic without analysis, following the natural intuition of the human mind, which without doubt is so clear on this point that it can be admitted without hesitation. Hence to discover and then solve the problem latent here is a purely philosophical concern; mathematicians rightly bypass it. Already in the introduction (p. 508) we find Poincaré exclaiming, "Who will doubt Arithmetic?" And that the same is not said about geometry comes solely from the problem of exactitude, not from the problem of necessity: "intuition cannot give exactitude (la rigeur)"; Poincaré said the same about geometrical intuition.

In one case, however, that of the critique of Euclid's Postulate V, from which non-Euclidian geometry has resulted, some seem to say that there is no necessity in geometry because Euclidian geometry is not the only one possible and therefore not necessary. But on closer examination, what they mean to say is only that classical geometry had not foreseen all cases; but in each case there is necessity. Those who then deem geometry as applied to natural bodies to be equivalent to physics, by their words *deny* the enormous difference that, as explained in the first chapter, exists between each science as to necessity; but in fact they rather *ignore* it.

Mathematicians, then, do not usually attend to the problem of necessity, and for that reason have more serious difficulties concerning what relate to the problem of exactitude. We say they have more serious difficulties, which does not mean that they all clearly distinguish these difficuties. There may indeed be a few who clearly understand what the problem precisely consists of: how and where the transition from inaccurate sense data to the accurate notions and propositions of classical geometry is possible.

Exactly, but briefly, we saw it already described in the introduction (p. 507 f.) by Kline and Poincaré; there also (p. 509) was mentioned the opinion of Study, who called the reconstruction of classical geometry, independent of modern analysis, a "dream" (eine Utopie). The same author, in the same book,[10] clearly sets forth the problem but cannot solve it except by considering geometry as a complex of hypotheses whose applicatiliby to the real world is to be examined, in entirely the same way as physicists do regarding their hypotheses. In doing so, he finally delcares the problem to be unsolvable.

No one, perhaps, has dealt with this problem more fully and more in detail than J. Wellstein in his book, Weber-Wellstein, *Enzyklopaedie der Elementar-Mathematik II* (3rd ed., 1925). We will also set out his difficulties on particular points; but for now, let us hear a good explanation of the general objection. After relating that modern criticism of the foundations of geometry in the 18th and 19th centuries begins from a consideration of Postulate V of Euclid, he

[10] E. Study, *Die realistische Weltansicht und die Lehre vom Raume*, p. 74 ff.

continues, "the development of the modern theory of functions has shown that the criticism of the foundations ought to have begun, not from Postulate V but immediately from [Euclid's] first definition: "a point is that which has no parts": σημῖόν ἐστιν οὗ μέρος οὐθέν.[11]

These words clearly state the problem of exactitude; for this modern theory of functions is opposed to the previous theory on this very point. For, as has been stated in the introduction, the earlier analyis "a continuum of real numbers", which analysis needs, geometers deduced from a consideration of the continuum; in this theory each real number corresponds to each geometric point on a line (whose existence was determined geometrically); numbers were defined by "coordinates" of points, i.e., by their distances (i.e., the proportions of distances to the unity of length). Modern analysis wanted to constitute this continuum of real numbers indepedently of the intuition (by the senses) of the geometric continuum, on account of the inexactness of sensitive intuition; and therefore it begins from a series of whole numbers, which is also sensitively exact; and from these it constructed (so it thought) real numbers purely arithmetically. This inexactness of sense intuition seems to weaken the first notion described by Euclid, the notion of the point; now there is at stake the very existence of a mathematical point, and the reality of the notion that denies all parts to a point and gives it position. Then the same inexactness is said to infect all other geometrical notions.

3. Thus in fact the problem of geometrical exactitude was once again declared unsolvable. But this historical fact was not thereby abolished: that for twenty centuries and longer these *notions*, and indeed as exact, were being entertained in the human mind, were present in the human mind. From extended being, as extended, the human intellect drew these notions and perfectly distinguished them from the smallest perceptible entities. Hence there can be no question about the very presence of these notions in the human mind.

The only question can be the following: do objects such as points, lines, and surfaces, strictly so called, actually exist exactly corresponding to these *notions*? And, because we are dealing with the essence of things, this is not the primary question, whether they actually exist in the natural world, but rather another question: does the extended, as extended, admit of exact limits, whether they *can* exist and how, whether "*esse* is proper to them," and how.

In this, therefore, the first question becomes: can objects such as point, lines, and surfaces be found in real extended being; then one must ask, can lines be perfectly straight, or perfectly circular; can surfaces be perfectly flat, or perfectly spherical; can there be figures that are perfectly equal to each other; can there be proportions of perfectly determined magnitudes? And so on.

Clearly, this problem entails questions about the existence of notions and of nexuses which are affirmed in judgements. Also, notions that are especially complex already presuppose judgements about the nexuses between simple notions.

[11] Op. cit., p. 9.

This question, therefore, which is purely a mathematico-philosophical question, is connected with another question, yet one distinct from it, namely the question whether there exist in this world, in this nature of things, such exact objects, whether there are in fact straight lines, perfect spheres, congruent figures, and so on. We will deal with this question even more frequently.

§ 2. The Prime Notions of Classical Geometry

1. The prime notions of exact geometry, which we must examine, are point, line, and surface. In §1 we have already heard Wellstein saying that criticism of the foundations of geometry must begin here.

He himself (op. cit., pp. 9-11) proceeds in this way, to conclude that these notions cannot legitimtely be derived from the experience of the senses. If we want to begin from these notions we must – he says – *postulate* their objective existence, indeed by an act of the will. In that case the transition from the inexactness of sense experience to exact geometry would be illegitimate, based on a decree of the will and not on the intuition of the intellect. Our problem that will pose the question about the manner of that transition would be unsolvable.

2. POINT, LINE, SURFACE

Regarding the notion of point, Wellstein proceeds in this way. He says that the notion arises through progression to the limit (Grenzprosesz), i.e., through an act of the mind which puts an end to the series of phantasms that is unlimited in itself. We begin, for example, from an intuition (sensitive) of a grain of sand which we imagine becoming ever smaller. From this there arises, and indeed in an increasingly determinate way, the imagination of a place ("Ort") in space, no longer having any dimension. But, he says, this process in the imagination quickly ends, that is, when the smallest imaginable is reached. From here to the end the matter remains obscure to us; by seeing or imagining we cannot follow the process of diminution any further. This is well said. But he goes on: it is unthinkable ["undenkbar"] that this process should end. Here he now appeals, and does so very well (although not logically in his own position) to the intellect, and indeed as being exact. For we understand that a process of diminution has no end if it proceeds by *dividing* the extended; one must add: if the diminution is made in a continuous *motion* (Artistotle's motion of diminution), it will indeed have an end. However, in both cases must be admitted what follows, according to Wellstein, namely: we must believe or postulate that there is in this process an end beyond which no further progress is possible and which it neverless fails to reach (in the supposition of a process of division). We approve these words, and even the words: "we must believe or postulate that this end, this limit, exists"; but we add: if we had nothing besides the method that Wellstein describes, if we did not reach an understanding of the existence of an invisible limit by any other way. But we do have an inellective intuition of this limit in another way, as we shall soon see.

If, however, Wellstein proceeds beyond certain clarifications, we must postulate the existence of a point in this way, that is, by a pure act of the will, not the intellect.

And from his own analysis he concludes: to Euclid's first definition must be added a postulate, by which one asks: do points *exist*? If we attend to the nature of the postulate, as here described, the very first notion in Euclidian geometry would be the offspring of an act of the will, not the intellect, which would mean the ruin of geometry, in its classical interpretation. An intellectual transition from inaccurate sense data to exact geometrical notions is declared impossible from the very first step, and our problem of exactitude has no solution.

In a way similar to the way in which the notion of point is derived by Wellstein from a tiny body, a grain of sand, are the notions of line and surface deduced. A line is seen as a thin thread and a surface as a thin leaf which becomes indefinitely thinner, and the limit of this process will be the geometric line and the geometric surface respectively. And in these cases the same difficulties will be encountered.

Actually, if only in this way could exact geometric notions emerge, the matter would be settled; but if the existence of indivisible limits is understood in another way, these progressions to a limit are entirely legitimate. The existence of their limits is no longer something to be *postulated*; we *understand* that the objects of these exact notions in extended being do exist. In fact we have this understanding, as will become clear from what follows.

3. Division of Extended Being (A Solid)

In order to discover this understanding, we do not begin from a point but from that which is the prime datum in this whole matter: from extended being as extended, not from a small body but from a body of any size; from extended being that immediately will manifest itself as extended in three dimentsions, as a "mass", a "solid."

The first "property" in extended being that we immediately discern by the intuition of the intellect looking into the phantasm is its divisibility. Let this extended then be divided in two parts, but only divided, let the parts not be moved or separated from each other. Now the parts are mutually *limited*; in this sense, but only so far in this sense, *a limit exists between them*. Again, it is immediately clear, that *through* the limit a *transition* can be made from one part to the other. And, let us note well: it is clear that the transition is made *in an instant*;[12] this last point is not clear to the imagination, which cannot perceive the indivisible, but it is clear to the intellect; *this limit between parts, therefore, as providing the transition, is indivisible.*

We can add certain aspects that better describe this matter, but now this intellectual intuition brings to light all that we are inquiring about. The transition from one part to the other takes place in an instant; therefore between the parts, inasmuch as they are mutually limited,

[12] 'In an instant, instantaneoulsy.' Latin, *in indivisibili,* not extended in space or in time; not admitting of more or less. Tr.

there is an opposition like that between things that are *contradictorily* opposed. When something which transits (it can be that by which one part touches another) transits from one part, i.e., it is no longer in it, by that very fact it is in the other; there is no extended between the two parts; and so on.

SURFACES.

Here, then, the limit is the surface and it can be described as "that through which there is a transition from one part to another." And then, precisely under the aspect under which the transition is made through it, it is indivisible. And thus the first indivisible, exactly indivisible (a prime exact geometric notion) we find intellectually in the phantasm in which this exactitude is lacking and now is understood to be lacking; we find it with the same certitude with which we gathered the *necessity* of its divisibility in the same phantasm of extended being. In the same act, if we are attentive, we attain both the necessity and the exactitude of this first geometric judgment, although neither its necessity nor its exactitude is present in sense knowledge; *in both respects* the intellect transcends the senses. Either one can serve to demonstrate the difference between the *phantasm,* which contains neither the necessity (but only the fact) nor the exactitude, on the one hand, and the *idea* on the other; and the difference with respect to exactitude is perhaps clearer to beginners than is the difference with respect to necesssity.

There exists, therefore, an inidivisible limit (indivisble under the precise aspect under which it is a limit: that through which there is a transition) in extended being, and it emerges in actuality through its division.

This limit is the surface; it is that into which ("against which") we inquire by looking at a limited body.[13] Not that it is always a plane surface, not to say polished, but as a limit it is everywhere indivisible. But in other respects (according to breadth and length) it is clearly manifest to us as extended, divisible.

LINE AND POINT.

With regard to this divisibility of a surface, a consideration similar to the one given above can be repeated here. A limit between the parts of a surface is again "that through which there is a transition from one part of a surface to another." This transition takes place again instantaneously; this limit, indivisible as limit, is a line, without a curve or anything else.

In another respect a line (inasmuch as it is not that through which there is a transition) is here again extended, divisible. And from its division there emerges a point, which is the limit of

[13] Someone may wish to get the idea of surface from this inquiry into the surface of a limited body, and indeed such that in the very sensing it would be indivisible as the limit of a body. For us, this is not enough; we hold that in the sensing of a surface we already see some depth of a body, so that in this sensing we possess the notion of the third dimension. If this were not true, our method of intellectual intuition would have to be used in the case of the line and the point, as we shall immediately indicate; for a line is certainly not without latitude in our sensing of it.

a line, through which there is a transition from one part of the line to another, once again in an instant. But now at last — it is once again clear that it is by the pure scanning of the phantasm by the intellect — this process of division comes to an end and we have arrived at that which is indivisible in every respect: the geometric point, what Euclid described as οὖ μέρος οὐθέν, "having no parts." It is not nothing: it is the limit of a line, which is the limit of a surface, which is the limit of an original extended being given to us immediately by external sense. Because a point is such a limit it is therefore not only something indivisible (in every way) but is "an indivisible having position," which is the standard definition for Aristotle and Aquinas.

The mathematical existence of these limits is clear; they exist in potency within extended and undivided being; they exist in act through the division of an extended being. And indeed surface emerges first, then line, then point. This is the sequence necessary from the very nature of extended being as extended.

If someone, in order to conceive the idea of line or point, wishes to follow the procedure described by Wellstein, that is, if one wishes to begin from imagining a grain of sand (or thin thread, respectively) and follows the process of diminution in this object, that can now be done without difficulty; for now one knows that there *exists* a limit which we approach by this process. And now the process described by Wellstein will be legitimate; no longer is the existence of limit *postulated*; and so in fact we have arrived at the notion of point and line.

INDIVISIBLES IN THE NATURAL WORLD

Do these indivisibles exist in nature? Let us talk about surface first; here the matter is immediately clear. There exist bodies that are limited and their limit is the surface. Not that there is any need for the surface to be geometricaly simple – a plane, a perfect spherical surface, and so on. Perhaps those that are found in the natural world are such that they do not have a geometrical name; perhaps they have very complex shapes, or perhaps they are very rugged, so microscopic as to escape detection. But they are truly "surfaces", which limit bodies; the transition through these limits, from not-this-body to this body, takes place in an indivisble moment. Can there exist in nature those surfaces, geometrically simple, that geometers are generally concerned with? The aswer depends primarily on the question whether these exist mathematically, whether they are possible in an extended being as extended; later, this question will be answered affirmatively, from which it will automatically follow that these surfaces will be possible even in the natural world.

Do lines in fact exist in nature? A hypothetical answer is quite easy: if limited surfaces exist, their limits, that is, lines, necessarily exist. But will that fulfil the condition? It is not a priori necessary; all bodies can be limited by their surfaces, which are like the limit of a perfect sphere. Here, then, is a surface that is finite, yet not limited. *It is* the limit itself, namely, the limit of a body (sphere) but which does not have in itself a limit to divide it. Here no line actually exists. If all bodies were so limited, no line would in fact exist in nature. But mathematically, i.e., as extended, those surfices admit of division and, if that is so, then indeed there will be lines in actual fact.

A similar question about the actual existence of points can be raised and be answered in a similar way.

4. DIMENSIONS

We have been discussing "the aspect under which a surface is indivisible," about "aspects under which it can be divided"; likewise in the case of a line relative to the surface of which it is a line. To these aspects correspond the "dimensions," as they are usually called. What we discover in the process of division described above we truly discover in our mind in considering extended being, that extended being whose notion we derive directly through abstraction from the phantasm is this, that the process of division can be applied to that extended being in three ways; first, to that being itself, then to the limit that results from the first division, and finally to the limit that is had from the second division. But the limit that is the result of the third division can no longer be subjected to this operation: it is indivisible in every respect, it is a geometric point. Here indeed is what we clearly behold in an intellective intuition into our phantasm. That whose limit is a point is a line, *extended in one dimension*; that whose limit is a line is a surface, *extended in two dimensions;* that whose limit is a surface, which is the original extended datum, will be *extended in three dimensions.*

What a dimension is cannot be defined according to a technical definition consisting of a genus and specific difference. Still, this notion is perfectly clear; what it is is perfectly well understood through the process described above. Extended being which has three dimensions is an extended that admits of that threefold division; and in this extended being there can be limits that have two dimensions, or one or none, respectively.

Here is the criterion of clarity for this notion: we to ask, can there be extended beings which, for example, have two and a half dimensions? The question makes no sense; and what we immediately see comes from the fact that the notion of dimension is perfectly clear.

Here is what we have been saying above: in extended being the following limits exist mathematicaally (in act or in potency), namely surface, line, point; and consequently they can exist in nature. This means: these beings having no or one or two dimensions exist *as limits*; it does not follow from this that they can exist separately. There are three-dimensional beings: body, extended substances. *Can there be* substantial beings that are two-dimensional or one-dimensional or having no dimension (this latter, of course, would belong to the extended world as "having position")? This certainly does not follow from what we have said above, from which we know that these ends exist as limits. As to the question whether such beings, having at least one or two dimensions, can exist separately, this writer can only reply by confessing his total ignorance.

Can there exist a being of four dimensions, whose limit would be a three-dimensional being? Once again I gladly reply that I have absolutely no well-founded answer to give to this question. I can only admire that there are mathematicians who answer in the affirmative without a scruple and yet who in general, when the question about the existence of mathematics is raised or about the possibility of a certain figure, are so critical. See *Cosmologia*, pp. 447-50.

5. *Regarding the objection drawn from the Moebius strip.* Our method of examining the notions of surface, line, and point seems to be natural to the human mind; and very often expressions are found that define these objects as limits. But some authors have raised an objection against this method, an objection that they deem decisive; accordingly, it merits our consideration. It is taken from that surface called the "Moebius strip" that we have already discussed in the second article.[14]

This surface has a marvelous property: it is said to be *unilateral*. What this means will be clear to one who looks at the physical image whose construction we described in that chapter, Let us draw a line on a sheet of paper in the middle of the sheet and parallel to its margin; by drawing this line we finally arrivce at the point from which we began, and lo and behold: the line is found on both sides of our sheet, yet without our having passed across the margin. We have remained, therefore, on the same side, and the sheet seems to have only one side. If we do the same on a cylindrical sheet, a line is drawn on one side only, and the other remains blank. Hence a Moebius strip is said to be unilateral, and is only apparently bilateral like a cylindrical sheet. To understand all this, one must by all means watch these moves actually being performed; yet, as we saw in the first chapter, they can manifest to the intellect an intuition into the necessity of this property. It is indeed a curious property.

See how from this property of surfaces, which are said to be unilateral, an objection is raised against our method of defining surfaces, which considers a surface to be as a limit among the parts of a three-dimensional extended object.

M. Couturat states the following: "Some authors define surface as that which limits a solid. But there exist surfaces that have only one side, or whose two sides continually pass into each other, so that they do not divide the space into two separate regions, *and consequently cannot serve to limit a solid*."

Wellstein writes in a similar way: "Forming the concept of surface solely from an external surface or from that which has a common beginning from two bodies, does not suffice, because there are surfaces that cannot, as a whole, limit bodies, and which cannot be surfaces limiting two bodies."

[14] *Gregorianum* 20 (1939) 37 -39.

They contend, therefore, that such a unilateral surface does not fall under our definition of surface (a limit between two parts of a solid extended being) because by the very fact of being unilateral they cannot limit solids.

Yet they are wrong. In order that the notion of surface (a limit through which a transition takes place in an instant) be verified and that the exactitude of the notion (of indivisibility) be manifest, it is sufficient to take any part of the Moebius strip; where this part is present (let us take the sheet between two fingers, which are separated from each other by this part) there is obviously a limit through which, as the intellect intuits, there occurs an instantaneous transition; nothing else is needed.

This solution of the difficulty, simple but decisive, is already indicated by A. Voss, as follows: "Unilateral surfaces do exist that limit no part of space; but this property belongs to them only on account of a special connection, while elementary parts maintain their character."

Another consideration of the Moebius strip (even of the whole strip, i.e., as its connection, *Zusammenhang*, is preserved and one side is continuous with the other) would have its importance; but we omit it, because what has been said suffices for solving the objection that is brought against our method of discovering and defining surface.

6. What This Contributes to a General Theory of Knowledge

Aristotle's Theory. From what we have said above we can gather the following. In extended being, as extended, there can be limits, which as limits are indivisible; this is known as *proper* to extended being, resulting necessarily from its nature. This property is not attained by sense knowledge, which lacks this exactitude. This exactitude cannot be affirmed about extended being as having *esse* in *sense* knowledge, whether external or internal; for there is no exactitude there, and the limits there are divisible, uncertain, not wholly determined. Exactitude, the indivisibility of limits, which is by understanding, can only be affirmed and must be affirmed of extended being, as having *esse* in itself, in reality, and is necessarily to be affirmed there.

And yet this knowledge, exact and of exactitude, is drawn by the intellect from the sensitive image by means of the intellect's abstraction from a phantasm, through the intellect's intuition in the phantasm. Also, in order for us *to judge* we need inspection into a phantasm. Because there is the transition that we described above (judgment), we are not enlightened unless and until we have looked into the phantasm. Only then are these judgements explicitly enuntiated, when this very epistemological question is studied (as we have done above). Indeed they seem to belong to those *virtual* judgments, as discussed in the previous article.[15]

As our consciousness testifies, the influence of a phantasm is required. In it we see and from it we draw what we affirm, either explicitly or implicitly or virtually. Hence it must be said

[15] Gregorianum 20 (1939) 45.

that not only the intuiting but also the abstraction by the agent intellect are present in this process. Aristotle's theory of abstraction, therefore, seems in every respect to be verified here. St Thomas has an excellent description of this in a well-known text,[16] where, speaking of the operation of the agent intellect, he says, "It cannot be said that sense knowledge is the total and complete cause of intellectual knowledge, but is rather in a way the matter of the cause." And in his response to the third objection in the same article he writes, "Sense knowledge is not the entire cause of intellectual knowledge and therefore it is not surprising that intellectual knowledge goes beyond sense knowledge." Both of these remarks square very well with what we find in our knowledge of these mathematical entities.

From the data of the senses, therefore, we arrive at an intellectual judgment of the indivisibility of limits; but what judgment affirms, it does not affirm about the extended as having *esse* in those sense data (it would not be true here, as we have said) but about the extended as having *esse* in themselves. This is to be affirmed and stated about all knowledge in general: we need the phantasm in order to abstract ideas and judge, but what we know first is not the phantasm but the thing; what is affirmed is not said about a phantasm but about a thing. Commenting on Aristotle's dictum in his *De Anima* III, 6, 431a 14-15,[17] St Thomas says:

> The comparison that Aristotle makes does not apply in all respects. For it is clear that the end of the faculty of sight is to know colors, while the end of the intellective faculty is not to know phantasms but to know the intelligible species[18] which it apprehends from and in phantasms, in accordance with the state of this present life. The comparison holds, therefore, as to what both faculties behold, but not as to that at which the condition of both powers terminates.[19]

According to this theory, in general what is affirmed in a judgment is not stated of the phantasm but of the thing that is represented in the phantasm. Our cases (about geometric limits) have this peculiarity (which does not hold universally): if what is affirmed in a judgment were predicated of the phantasm itself, it would be *false*. For our mind can through reflection return to its operations and consider the phantasm itself as an object of knowledge – just what we have been doing in our analysis. And the outcome of such reflection is what we have been saying: in the phantasm itself, and in sensation in general, there is not that exactitude that is affirmed in the judgment of the intellect concerning the limit of extended being; therefore it cannot truthfully affirm it of extended being as to its *esse* in the phantasm. What we affirm is valid concerning the extended *in itself*; what we assert about it is that which is present in the *idea* (composite) with exactitude.

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[16] *Summa Theologiae* I, q. 84, a. 6 at the end.

[17] "To the thinking mind images serve as if they were contents of perception."

[18] St Thomas, as is well known, did not intend to say that the intellect first knows its species but rather the species that are in things.

[19] *Summa Theologiae* III, q.11, a. 2, ad 1m.

PLATO'S THEORY. If our analysis confirms Aristotle's theory, other theories ought to be excluded by it. And in fact this is to be said about Plato's theory: 1) The function of sensation is not to prepare and call up memory. From sense data our mind truly derives the notion of the extended and, to consider it, it must recall the phantasm and simultaneously it grasps in it the exactitude of a limit. The way in which this happens has been described by no one better than by St Thomas: sense data are not the total and perfect cause (*ergo* only the partial and imperfect cause) of intellectual knowledge, but are rather, in a way, the matter of the cause. 2) What we grasp by our intellect in the phantasm clearly indicates in the extended as such, hence also in the physical extended, that there must be exact limits.

KANT'S THEORY. Even the Kanitan theory of subjective form seems to be peremptorially refuted by our analysis. For in that theory extension is not found in being in itself, unknown to us, but only in phenomenal being, which has no *esse* except in our sense knowledge. But according to this mode of being, the extended *lacks* the exactitude of the limits. Therefore exact intellectual geometric knowledge would *nowhere* find application: not in being in itself, because it lacks extension, nor in phenomenal being, because it lacks exactitude.

THE THEORY OF EMPIRICISM. From what we have seen, certain statements of Stuart Mill have now also been refuted. Recall his statements above on page 3: "There are no points without magnitude, no lines that lack breadth or are perfectly straight ... no circles whose radii are perfectly equal ... nor are these even possible." It is now obvious that these statements are false with regard to the existence (at least possible) of indivisibles: points, lines without breadth. Whether the same is to be said about qualitative elements and about equality (about lines perfectly straight and perfectly equal radii of a circle), which he denies, it is not yet clear. That pertains to the second part of the problem of exactitude, whose solution depends on another means of critical inquiry; for this reason we said above that Mill confuses different aspects of exactitude.

It is also false what we hear from this same author: "I conceive our idea of a point to be the idea of the *minimum visibile*." For in our analysis we find the notion of exact limit, which leads to the notion of an exact point, and at the same time by judging we find the mathematical existence of these exact notions.

NOTE. We *began* our analysis *with that point* in which we first discover the *intellective* intuition into extended being: into its divisibility, exact limitation, and triple divisibility corresponding to the three dimensions. In the analysis we discover the *intelligibility* of this extended being, for what we affirm we know as a property flowing *necessarily*, therefore intelligibly, from that which is extended. Hence – every intelligible is being, *esse* belongs to it – from it and in that moment we know: this extended is a being, to it *esse* belongs. So there is no need to inquire about the origin of sense data.

How sense knowledge, in which we find this intelligibility, originates, and in particular how we arrive at sense knowledge of the extended, which then reveals itself to be tridimensional, is more of a psychological question. A theory of the intellectual knowledge of that knowledge (which is the science of geometry) begins with that intellectual intuition.

We have already touched upon that opinion which would find in sense knowledge surface without depth, extended in two dimensions. To us, this does not seem correct. We are of the opinion that in the first look at a "surface" there is already present the aspect of a certain depth. But since this question belongs with what precede that first intellective intuition, we shall not deal with it here. At any rate, if this opinion were true, our analysis would still remain intact in terms of the line and the point; nor would it be false with regard to the existence of an exact surface.

7. DIVISIBILITY TO INFINITY

A. DIVISIBILITY INTO EXTENDEDS. The notion of indivisibles (of exactitude, therefore) allows us to enlarge upon what flow from a consideration of necessity alone with respect to the divisibility of the extended.

For from this we already know: an extended can be divided into two parts; there is no need that these parts be equal (that would involve the element of exactitude), but they are extended, just as the whole was extended. This is understood from inspecting the phantasm of any extended, e.g., a line (inexact, of course). In this phantasm we intuit by our intellect that divisibility flows as a "property" from the nature of an extended as extended. Moreover, we intuit that the parts have the same nature of extension as the whole. The notion of 'extended' is as fully verified in them as in the whole of which they are parts. And these in turn, therefore, can be divided into two smaller extended parts. Our imagination can continue this way only to a certain limit, that is, up to a certain *minimum imaginabile* (which is not determined exactly). But in these parts also the note of 'extended' is verified *univocally* (together with the original whole). And since "can be divided into extended parts" flows from the nature of the extended, the same holds for the imaginable minimum, and it likewise holds for its parts: to infinity.

From this we have the following. From that one phantasm, from which we began, by looking at or "observing" it, we are able to understand that *every* extended (line) is divisible into two smaller extended lines; "every", that is, the principle is valid for the whole genus; it expresses a proper feature of the extended as extended. Next: every extended, because it is extended, is divisible into two smaller *extended parts* – to infinity. We grasp this in the phantasm, but by transcending the phantasm. Not, of course, in that exact points have now been found – there was no discussion about points – but in that we understand that there are smaller extended parts that are *quam minimum imaginabile* – as small as can be imagined. And this "dichotomy" can proceed to infinity; thus there is always something that remains to be further divided.

This principle of divisibility to infinity can be taken more strictly with the help of exact elements (points). Through dichotomy (and also through trichotomy and so on) an extended indeed cannot be divided to the extent that only indivisibles result. It can be asked: could not this be done by a single division, not successively, but one that in a single action would exhaust its divisibility, leaving nothing but indivisibles? This was not settled by what we have said so far, but a negative answer is clear on other grounds.

Consider this: if the parts into which an extended is divided are added to each other they again constitute an extended whole that is equal to the entire original whole. Hence: an extended whole can be composed of those parts into which it can be divided. Indivisibles however, points, lack all extension, they are "of no extension"; and adding nothing to nothing cannot result in something; from the added points, nothing extended can result. Indeed it must be said that *geometrical* addition, in the case of lines that are added to each other, makes perfect sense; in the case of points it makes no sense at all. Nothing cannot be added to nothing, but only quantity to a quantity. (The arithmetical addition of points does indeed make sense: we can think of one, two, three, etc., points.)

If, however, an extended cannot be composed of points, neither can it be divided into them as into parts. Hence it follows that *in whatever way* an extended is divided (not only by dichotomy), it is *always* divided into parts that are themselves extended, which can again be divided; and so this universal principle must be established: it is divisible to infinity. (On the importance of this divisibility in the philosophical history of the continuum, see *Cosmologia*, pp. 22-40.)

B. THE LINE AS A "COLLECTION" OF POINTS. From what we have been saying, there follow a number of things that can be worthy of note. It was obvious that a line cannot be made up of points; hence a fortiori it will not be able to be a collection of points, if the word "collection" is understood in the ordinary sense, that is, such that a line would result *solely* from the addition or position of points. We say "a fortiori", because a line is continuous (as distingushed from a series of contiguous items), and therefore is intrinsically one, and this pure collection would be but a series of contiguous items. But this must be said above all: if intrinsic unity is solved, a line cannot be resolved into a collection of points.

Already in ancient geometry the line was spoken of as a "locus of points" which have a definite propriety; thus the bisector of an angle is the locus of points equidistant from its sides. By this expression we do not intend to describe that line as a collection of points; rather, the meaning is that every point equidistant from the sides of the angle is situated on that line, and vice versa, that every point in that line is equidistant from the sides of the angle. A bisector is considered only as the "matrix" of those points, the potency (matter) out of which and in which equidistant points can be actuated.

The situation is otherwise in the modern theory of collections which has been constructed by Cantor, at least if it is conceived in the way in which its author intended it. There, in fact, if a line was spoken of as a “collection of points”, they understood it as a pure series of contiguous points. In a similar way, other collections, whose individuals (elements) had been determined solely by definition of *genus* and *species*, now were being considered as made up of a multitude (generally infinite in act) of *individual* data determinate in themselves. This ideed, according to what we have explained above, cannot be admitted. But this naïve theory leads to contradictions (“paradoxes”). Hence by most mathematicians it is used only in its corrected form, in which, given the generic or specific definition (given the law of the formation of a collection), the individual elements are not yet considered as determinate in themselves. Such a collection verifies Aristotle’s notion of potency, which (for individuals) is to be actuated by further determination. In such a theory a line is now not a collection, that is, a series, of points, but is a potency in which points can be actuated, as Brouwer says: “a matrix of points to be thought about simultaneously”.[20] And it would be worth while if a mathematician well acquainted with Aristotle’s theory of potency were to develop a theory of collections, something of great importance, if seen in the light of this Aristotelian theory.

As we have said, most people now do not accept the naïve theory of collections, but in instead more or less correct the theory. Up to now it seems to be fully accepted by B. Russell, who thought that by it he could solve the classic antinomies of a real continuum.[21] He was wrong, as is clear from what we have said. Mathematicians who correct the naïve theory of collections – most of them, at least, so it seems – think that its contradictions come from the fact that it admits actual infinity. So Poincaré says, “*There is no such thing as an actual infinite*; the Cantorians have forgotten this, and they have fallen into a contradiction.”[22]

Whatever it is, an extended cannot be considered as a collection of points, resulting from their addition or position. It could be considered as matter, potency, from which and in which points can be generated to infinity. More often in mathematical explanations the extended or figures are described as collections of points. One must always note whether it is taken in the first sense or in the second; if in the first, there is danger of error; and, even if errors are avoided, nevertheless such an explanation cannot be justified philosophically. This also (not only) obtains in the case of equations in analytic geometry, as we note in the following.

A NOTE ON EQUATIONS IN ANALYTIC GEOMETRY.

Such a formula indicates the relation among the coordinates that determine points that are located, for example, on a line. This line, if defined otherwise, will be the classic “locus” of points that have a determinate property, and this property is expressed in a formula. If this formula is used to define a line, there can be a danger that the line may be considered as a

[20] *Over de grondslagen der wiskunde* (1907) p. 8

[21] See *Revue de Métaphysique et de Mor*., (1911), p. 183. See also *Cosmologia*, p. 432

[22] *Science et Méthode*, p. 212. (Poincaré’s emphasis. More on this in *Cosmologia*, pp. 431-35.)

collection of points in the incorrect sense. For examaple, take the equation $y = \sin x$. This determines points whose "ordinate" y through the formula indicated can be calculated by a chosen arbitrary value of the "asbscissa" of x. To each abscissa there corresponds one ordinate; the choice of the abscissa determines the ordinate and consequently the point. If one interprets it this way, that a line can be made up of points so determined, one is considering a line to be a collection, i.e., a series, of contiguous points. But no line can come to be in this way. Therefore another interpretation must be sought; here is one explanation that is often given by mathematicians. To wit: To an abscissa (x) is assigned a *continuous* variation which it can have only through motion, not through the successive choice of determinate values; therefore let the line perpendicular to the axis of abscissae so move by continuous motion in accordance with a determinate law and let it depend on time alone as an independent variable: $x = f(t)$. Let a point move on this very line, again with a continuous motion which is determined by its own law and depends solely on the same time; the distance of a point from the line of abscissae always indicates the ordinate: $y = F(t)$. From the composition of these motions there results a continuous motion which generates the line whose equation can be found by elimitating the variable t from the two equations: $x = f(t)$ and $y = F(t)$. In our case the two motions can simply be expressed by the equations $x = t$ and $y = \sin t$. And the geometrical meaning now is: the line (of ordinates) moves; always perpendicular to the axis of abscissae (x) in a uniform motion ($x = t$); at the same time a point in that line moves so that its distance from the axis of abscissae is expressed by the law $y = \sin t$; it moves therefore in a straight line, ascending and descending in accordance with the periodic movement. And from these two motions there results the composite motion of the point, which describes the line that corresponds to the equation $y = \sin x$. The resultant line is not a collection, i.e., a series, of points, but is continuous because of the continuity of motion.

Application. This can be applied to solving the argument that many modern mathematicians have raised *against* the validity of intuition, from which classical geometry arose. The argument is taken from functions, which are indeed analytically continuous but do not have a "derivative" (difference quotient) either for one value of an independent variable or even for all. Continuous curves geometrically correspond to continuous functions; to a "derived" on one point there corresponds a tangent to the curve on that point. To these functions, therefore, curves correspond, which would indeed be continuous everywhere, but they would lack a tangent on one point, or even on individual points. But intuitive imagination, which is classically the root of intellective intuition, cannot represent to itself a continuous curve that does not have a tangent everywhere; and, if the intellect is to go beyond this imagination to exact lines, it must affirm the following: there is no continuous line that does not have a tangent everywhere. But, they say, exact analysis demonstrates that there are such lines, and intuition, therefore, leads to error and should no longer be given credence.

At this point it could be worth while to examine the argument made by the eminent French mathematician Henri Poincaré in his *La valeur de la science*, pp. 17 ff.:

> As we notice more and more, intuition cannot give us rigor or even certitude. Let us quote some examples. We know that there exist continuous unexpected derived functions. Nothing is more shocking for intuition than this proposition which logic has imposed upon us. Our fathers would not have failed to say, “It is obvious that every function continues to a derivative, since every curve has a tangent.” How can intuition deceive us at this point? It is when we try to imagine a curve, and we cannot represent it to ourselves without thickness; likewise, when we represent to ourselves a straight line, we see it in the form of a rectilinear string of a certain size. We know full well that these lines have no thickness; we force ourselves to imagine them as ever thinner and thinner and so to approach a limit; we are getting there to a certain extent, but we shall never reach that limit. And so it is clear that we will always be able in this way to represent to ourselves these two straight ribbons, one rectilinear, the other curvilinear, in such a position that they lightly touch each other without encroaching on each other. Thus we shall be led, at least having been informed by rigorous analysis, to conclude that a curve always has a tangent.
>
> The argument that we are examining is summed up in this last line.

Let us take the simplest example, a classic example, of such a function that is continuous everywhere but in one point lacks a derivative. Therefore, they say, the curve that corresponds to this function would indeed be continuous, but at one point would lack a tangent. And therefore it would contradict intuition. Take the function $y = \sin l/x$. It is a periodic function, like the function $y = \sin x$, but its “sinuosities”, if we go to the point of origin of the coordinates ($x= 0$), ($y = 0$), *first* have an ever smaller amplitude (the distance of the greatest and the least from the axis of the abscissae), because the function of a bend is multiplied by x and x always approaches zero; and *secondly*, “sinuosities” are ever more and more compressed together, so to speak, because each period corresponds to the variation which equals $2\ \pi$, and this variation now reaches through it the minor variation in x by which the value x is now less and therefore l/x is greater. Towards the origin, “sinuousities” are compressed to infinity. But this function in itself has no meaning; therefore there is added, and this is resonable, to the above equation this system of values: if $x = 0$, $y = 0$.

However, a function that is so definite is everywhere continuous; and yet in its very origin it does not have a derivative. Hence they conclude: a geometric curve that is defined by those equations has no tangent in its point of origin, although it is continuous here also.

But we ask: does this cruve really “exist” geometrically? We will try to construct it by the composite motion of a point described above. Let us say then that $x = a - t$; then we have $y = (a - t) \sin l/a - t$. As soon as $t = 0$, the movable line is to the right of its origin in distance a: as soon as $t = a$ it will be on the point of origin and the point of the curve will be $x = 0$ and $y = 0$. Can this point be reached in such a way that the motion of the point on a moveable line of ordinates is actually realized according to the law $y = (a - t) \sin l/a - t$? In order for the origin to be reached, the value of this function actually-infinitely would have to (periodically) reach the maximum and minimum, a movable point on a movable line would have to ascend and descend in it actually-infinitely. But – even if one holds that an actual infinite is not self-contradictory – still, the infinite by definition *cannot be transgressed.* Thus: the motion of a point on a mobile line of ordinates, if it has to reach the origin, is impossible; therefore it is not clear that this curve according to this method "exists" geometrically. If our line continues only to a certain distance, even a very small distance, from the origin of the coordinates, there is nothing impossible here. Therefore: any part of a curve exists, so long as it is not thought to have been extended to its origin; but in such parts it has a tangent everywhere.

Whether this function has any sense analytically is not being examined here; but to this equation there corresponds no total geometric curve arising from motion. And they do not indicate any other method for constructing a curve. Therefore this is what follows from intuition: that every curve in any point in which it is continuous has a tangent is not refuted by this example; so far, indeed, it is rather confirmed.

Afterwards, first by Weierstrasz, functions were constructed which are even more marvelous: these are continuous everywhere and nowhere have a derivative. To them are said to correspond curves that are everywhere continuous and yet nowhere have a tangent. This difficulty will have to be solved in the same way as above: those curves (kritzelige Kurven) are "infinitely sinuous" so that on the same account they do not exist geometrically. Curves are constructed "approximately" but they are normal, that is, they have tangents.

See the ingenious treatment of these curves by F. Klein in his *Anwendung der Differential und Integralrechnung* (later printing, 1907), p. 39 f., 51 f., 78 f., 83-102. This is also treated by O Becker in *Bieträge zur phaenomenologischen Begründung der Geometrie und ihrer physikalsichen Anwendungen*, in *Jarhbuch für Philos. und phaenomenolische Forschung* VI (1923) pp. 90-93. His solution to the difficulty is not everywhere clear; perhaps in reality it is the same as ours. The curve constructed by Weierstrasz is described by Brunschwieg, *Les étapes de la philosophie mathématique*, p. 338.

8. Geometry as a Constructive Science

In what we have said above, we have been looking at the simplest examples of geometric "construction", which manifest at the same time the mathematical existence of these objects. The extended parts of an extended whole were constructed through division; by that very fact surfaces, lines, points were constructed as exact limits of parts. Thus also were found certain principles, which are usually called "axioms of order." For divisibility to infinity, as found above, is equivalent to the principles:"between two points on a line there is *always* a third that is distinct from the first two"; hence it follows that between two points further points are found and so to infinity. That is, they are there in potency; they can be constructed, or actuated, and in this sense they without doubt "exist mathematically."

This notion of "construction" is described by Kant as mathematics *par excellence*; in being able to construct its own objects, the discipline of mathematics differs from others – from metaphysics, for example. There are many others who strongly emphasize this property of the science of mathematics, indeed occasionally exaggerate it with the result that they are led into error. For they attribute too much to the *activity* of the human mind; sometimes they attribute to it everything in this matter and there is talk of the "creation" of these objects by the human mind; and indeed in such a way that this notion of creation (for ideal objects, of course) is to be taken in the strict sense of the word. They speak, in fact, about free creation. Clearly, in this opinion the problem arises about the validity and applicability of such creation, and so the question becomes unsolvable.

This way of interpreting the construction of objects in mathematics is erroneous, no doubt, as is clear from what has been said. For in the simple cases which we have been examining so far, that is already obvious. True, we construct with our minds; but we construct out of preexisting matter, from extended being as extended. This is of greater importance: we *discover* that the extended is suitable matter for us out of which to construct. We discover the "properties", in the strict sense, of the matter which can be used for such or such construction. We discover first, if one may say so, the "constructibility" as the proper use of this subject-matter; we discover what can *be made* in and out of this matter, this potency. Hence it is clear how appropriately extended being is named by Aristotle and the scholastics: intelligible matter.

And in finding this, we undertand at the same time our dependence on this matter and on the data of the senses. We find *that* various figures can be constructed because we understand this aptitude in an extended being. *What* can be constructed: a multitude through divisibility, the exact limits of the parts, points on lines, and indeed to infinity – all this we discover by the intuition of the intellect as looks at experiential data, in the phantasm. On both parts we do not *impose*, rather we *receive*, we do not act except by abstracting and intuiting at the same time. Our acitivity in constructing is linked through the material out of which construction is done, through that which the intuition of the intellect *discovers* in looking at the phantasm. It is found in these simple cases, it also must be given attention to elsewheere, and it is worth while to consider it attentively. For our activity there is verified what we have previously heard from St Thomas: the data of the senses are not the total cause but the matter of the cause. And this connection with the real extended is not solved even in a moment.

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