ON THE PROBLEM OF NECESSITY IN GEOMETRY

As we have already said in our introduction, the common opinion of the human race - both learned and unlearned - up to the beginnings of the modern critique (of mathematics) was this: mathematical knowledge is derived from the data of sense-experience. Hence there arises for the philosopher the problem as to how this is to be explained. For the data of our sense-knowledge do not of themselves imply any <u>necessity</u>, but mathematical knowledge claims that it is necessary. Further: the sense-data which constitute the basis of geometrical knowledge (not the data which are presupposed in the elementary knowledge of arithmetic) lack <u>exactitude</u>; and nonetheless geometry, no less than arithmetic, presents itself as wholly exact.

By empiriciam (Stuart Mill) the problem is ultimately considered insoluble; consequently it denies exactitude as much as necessity and geometrical science becomes empirical in exactly the same way as the physical sciences; they are only approximative sciences and their alleged necessity is only an illusion which has its origin in custom.

Plato considers sense-data as the occasion and preparation for eliciting remembrance in our mind. Kant, who neglects the problem of exactitude, thinks up the subjective form of our sensibility.

Aristotle solves that problem by the theory of abstraction in which experience plays its part; but this solution, which was also that of S. Thomas and some of the earlier scholastics, seems to be forgotten.

In the critique of modern mathematicians scarcely any attention is paid (except in one case) to the problem of necessity, with all the more attention being given to the problem of exactitude. Later we will deal expressly with this latter problem, but first we will deal with the problem of necessity. The principal question of this section will be; can experience lead to necessary jugements; and if this is found to be true, how is it to be explained?

I. Some Examples from Arithmetic.

1. - On the Platonic Number 5040. In the fifth book of 'The Laws' (7373 - 738b) treating of the new state to be established, Plato selects the number 5040 as the most suitable number of inhabitants, farmers and soldiers:

"The farmers and those who fight for the state are to be 5040, because of the convenience of this number".

The convenience of the number consists in this, that it can be divided equally in so many ways (namely 59).

"First the whole number will be divided into two parts, then into three, into four, five and so on up to ten".

If one knows that 5040 = 7! (i.e. 2 x 3 x 4 x 5 x 6 x 7) what Plato says will immediately be clear. But moreover all the possible equal divisions of this number, i.e. 59, are known to Plato. For he continues (738a):

"But let us state that that number is most convenient which admits of the greatest number of divisions, and especially of successively ordered divisions. For not every number admits of all such divisions. But the number 5040 can be divided both for war, and for peace, for all meetings, societies, tributes, into not more than 59 parts from one to ten consecutively".

(According to the interpretation of Marsilius Ficinus)

2. - To apply this to our question. How can this number 59 of all the equal divisions be established? It is quite simple for us. Thus: 5040 = 2 x 2520 = 3 x 1680 and so on to the last division which results

in something new: namely = 70 x 72. For if we go on we get a repetition of what went before. If we make an integral table of these divisions and look at it, by counting we find that their number is 29 which contains 2 x 29 = 58 possible divisions into equal parts, to which one must be added: the division into 1540 units. Thus in this way we arrive at the knowledge Plato had, that the number 1540 admits of 59 divisions into equal parts; this knowledge is certainly a necessary judgment.

Let us attend first of all to this part of the process: when we have made the divisions we must count (calculate) their number; this counting is an experimental operation in exactly the same way as counting the number of men in this room is an experimental operation. This latter operation does indeed lead to a true judgment, but not to a necessary truth; the number of people present can vary. The difference arises from the fact that in Plato's case this experimental operation is applied in necessary matter, in the second case it is applied in contingent matter; but we also intuitively grasp that our operation of itself leads to a necessary result in both cases. Later we will enquire how this comes about; now it will suffice to accept the fact itself.

Let us note further: even before the calculation we know that the result will be necessary and in such a way that we already know that the number of divisions will necessarily be some determined number; the calculation only gives us the specific number.

3. - We can make a futher step. We found the possible divisions by applying the elementary, but nevertheless derived, rules of modern arithmetic. How Plato did it matters little; but it is of great importance to find out that it can also be done by experimental means, which require a long time but are nevertheless possible. Thus: We take 5040 symbols for numbers, or even small objects v.g. pebbles, from which the calculus gets its name. And we try to decide how this large number could be divided into equal parts. First whether into binary numbers, and to

decide this we dispose the pebbles thus:

1 3 5 - - - and in the end we get 5039
2 4 6 - - -

How many are there? By counting we get 2520.

We proceed to triads and we get:

1 4 7 - - - 5039
3 6 9 - - - 5040

We can proceed in the same way up to division into tens; but if we try to divide our number into parts each of which consists of 11 individuals, we discover this to be impossible (a necessary judgment!); for we get 458 such divisions, but two pebbles remain over. Thus, if we were unaware of it before, we now discover the necessary truth of Plato's assertion mentioned above: "for not every number admits of all divisions of itself".

If we go on patiently, as above, we will discover that 59 divisions into equal parts are possible; we will discover this truth as a necessary judgment. Nevertheless in the whole of this process we were only employing experimental operations, of such a kind however that each of them can be seen to involve absolute necessity; it is clear that such a necessary effect results from the operation employed, although what that effect is is only known empirically.

To show this more clearly let us compare this process with another partly similar partly different: if pebbles of different colours, some white and some black, are put into one jar and then we take a hundred out, of these v.g. twenty will be white, the rest black; if we repeat the operation the result will in general be different and both results will be contingent. But if we distribute the number of these pebbles into pairs the result will be, and be known to be, necessary; it will be either: this cannot be done, or: it is possible and the number of pairs is such and such. In one case we intuitively grasp the necessity, in the other the contingency of the effect of a similar experimental operation.

Later we will explore the source of the difference between these cases (and others also in which we have no intuitive grasp of necessity or contingency); let us now fix our attention on what we have discovered: not every experience leads to a contingent judgment; there is an experience which gives us necessary knowledge; and: from such an experience we not only derive the notions, but the necessary judgments themselves, the nexus itself between the terms of the judgment and its necessity.

Now merely in passing we note one fact: we spoke of the pebbles with which we performed the operations; we know that these are not necessary; the same effect will be obtained if we consider any clearly distinct elements, v.g. the letters a, or crosses, or the points which we write. We know intuitively that the result follows from the number of the individual things alone and in fact necessarily results from the nature of the number; and in this sense in each of the judgments the subject is the cause of the predicate, which consequently has a necessary nexus (connexion) with the subject. And already we begin to understand that Aristotle's theory of abstraction is here verified.

4. - On Prime Numbers. It will help to offer other examples. First we take what are called prime numbers and consider how many of them there are below a fixed number; v.g. below 100 there are 25 prime numbers, namely 2, 3, 5, 7, - - - 97 (here 1 is not counted as a number). The way of establishing this number must, at least in the beginning, be experimental. (At least in the beginning, in the first range of numbers. Later there are methods for reducing the method of determination for the higher ranges to known numbers from the lower range; but even there the experimental method could be employed). The determination of the prime numbers up to 100 can be done thus: a list is made of all the numbers and then the so-called "sieve of Eratosthenes" is employed; i.e. the compound numbers are expunged and those that remain, all these

and only these, are the prime numbers; their number is determined by experimental computation and is found to be 25. Moreover, a purely experimental method could be used to solve the question as to whether a determined number (v.g. 91 = 7 x 13) is a compound or prime number, and the whole process would be experimental. Nevertheless it would lead to a necessary judgment.

5. - On the beginnings of the Art of Combination.

We have a similar example in the elements of the art of combination. As is known, the number of the permutations of a larger number which consists of n elements is n! (the 'faculty' n). The first steps in the deduction of this necessary theorem are these. If we have two elements a and b, there are two permutations i.e. the series composed of these elements which only differ in order, namely ab and ba. If a third element c is added, this can obviously be added to each of these series either before the first element, or between the two, or after the second, and from the first permutation we get: cab, acb, abc; from the second: cba, bca, bac; neither more nor less. From three elements we have six permutations. But this double threefold combination becomes evident to us only by inspection either in the phantasm or in actual writing-down (in each case a spatial element also enters). We have again experimental sense-operations involving a concrete reality, which is the matter (material element) of our necessary intellectual intuition. And again it is clear that from experience there can issue judgments which extend our knowledge and which are at the same time necessary.

The number of these permutations will be 2 x 3; how v.g. by the aid of "the principle of complete arithmetical induction" the general theorem enuntiated above can be reached does not interest us here.

6. - Judgment on the Addition of Numbers: 7 + 5 = 12

As the source of such a judgment, which is certainly a necessary judgment, the same kind of experiment can, indeed must, be used. This can easily be established.

It can be used. Suppose we do not yet know what is the specific number which will result from the addition of a number comprising 5 units to a number with 7 units. There certainly was a time when we did not know this. And so we carry out the addition by an actual operation. We already have 7 individual objects: we add the other number to this successively until it is exhausted, counting at the same time v.g. with the help of the fingers of one hand. After the addition of the first number the result is 8, then 9 - - - when the number of these digits is complete we find 12. And then we elicit the necessary judgment: the number which results from the addition of 5 to 7 is the number 12. It is patently obvious that we can in this way find the answer to the question "What is the result of that precise addition?", or that we can find the predicate (previously unknown) of the necessary proposition "from the addition of 5 and 7 there results the number 12". But if that is possible we have another verification of the proposition: there is an experience which leads to a necessary judgment.

Indeed such an experience seems to be necessary in judgments on the addition of numbers; nor does, there seem to be any other method available to discover for the first time the number which results from a given addition. This can be confirmed by the consideration of two methods which have been used to deduce that judgment from definitions with the aid of other principles.

7. - Leibnitz' Method. Thus Leibnitz holds that that judgment is analytic in this sense, that it can be deduced from the definitions by means of the syllogism. He rightly observes that the definition of the number '2' is: the number which results from the addition of a unity to a unit; '3' is the number which results from the addition of a unit to 2; and so on. Each number is specifically defined by its origin from the number immediately preceding, by the addition of a unit, by "recurring

definition". We say: they are specifically defined, for the genus is already determined; for every number is a multiplicity; then they are specifically determined - and indeed necessarily, otherwise we would not be dealing with a species - by the method which recurs (to the unit) or by "arithmetical induction", so that '2' is first defined, then '3' and so on. The scholastics also share this view. In passing we may observe: number (two, three etc.) or the ordinal (second, third, etc.) is naturally known first the answer, according to what has been said above, would seem to be: in relation to the genus (multiplicity) the cardinal number is prior, in relation to the specific difference the ordinal number is prior.

Having thus established the method of the specific definition of each number, Leibnitz then reasons: From these definitions - hence analytically- it can be shown v.g. 2 + 2 = 4.

"Démonstration:

2 et 2 est 2 et 1 et 1 (définition 1)

2 et 1 et 1 est 3 et 1 (déf. 2)

3 et 1 est 4 (déf. 3)

Donc (par l'axiome) 2 et 2 est 4"

Hence such a judgment, expressing what results from the addition of numbers, is considered an analytic judgment in this sense: in its origin experience only plays a part insofar as from it the notions are derived, but the nexus between the notions, which are affirmed in the necessary judgments, are not derived from experience, but only from those notions (definitions) and syllogisms, purely analytically. However we have two observations to make and both will indicate the influence of experience on the knowledge of the nexus itself also.

The first is this. In order that the series of syllogisms be of the required number to insure that the final conclusion should be precisely v.g. 5 + 7 = 12, the individual members of this series will have to be counted, enumerated very carefully, for otherwise we will make an error. That is perhaps not evident at first sight if the series is very short, as in the example from Leibnitz; then there is certainly counting, but it escapes our notice. But if the series is longer, then it becomes clear that we must count. And ultimately the same experimental element which we have found above is present even in Leibnitz; we counted pebbles, Leibnitz syllogisms, or numbers in syllogisms; experience which uses the phantasm is not absent from this quasi-analytic deduction; but once again it is an experience which includes an intuition of necessity.

The second observation is this. Leibnitz very well explains the specific definition of numbers. Let us note again that here there is question of the definition of essences; hence as will be clear immediately, a certain knowledge is presupposed. Here is an example: suppose we have an urn full of small balls and by a blow some of them are thrown out; by counting we find the number is three. Can we now establish this definition: three is the number of balls which are thrown out of this urn by one blow? Clearly not; the next blow could throw out 4 or 5 balls; the necessity of the effect is lacking, it is contingent. But in the definition of an essence necessity is required. And so the way of defining successive numbers which Leibnitz uses in his argument, can be accepted, because this necessity is present and is pre-known by us. We knew that from the addition of a unit to a determined number, necessarily and so always, the same other determined number would follow, and not from 3 now 4, another time 5. Where did we get this knowledge? Only by directing our attention to the phantasm of the number to which we add the unit do we know the necessity of this judgment, that it is thus that the number determined in its essence originates. It is not a question, therefore, in those definitions of the notions only, but also of the judgment for the evidence of which we need a similar experimential medium. And this judgment is an essential - though not expressed -

element in Leibnitz' series of syllogisms. We will presently return to this point (and to the other pre-cognition).

8.- Mercier's Method. Mercier uses another though similar, method to prove the purely analytic character of our judgment: 7 + 5 = 12. He wisely notes that the concept of the subject, (7 + 5), is the concept of a sum (d'une somme), i.e. of what results from an addition, but not the concept of any sum but of a precisely determined sum, because it is the sum of two determined numbers. And he argues from the principle "the parts making up a whole and the whole are identical". Thus by the pure principle of contradiction he concludes to the identity of the subject and the predicate of our proposition, which are (7 + 5) and 12 respectively; and so we have our judgment.

Nevertheless we have not as yet got the knowledge of the species of this sum; we know that it is a specifically determined number, but that it is the number 12, we do not yet know after this argumentation. More is required for us to attain this knowledge. Hence "des représentations symboliques" not only can (as Mercier says) but simply must be used if we are to know the identity between the subject and the predicate which is 12. This is his procedure:

7 is written 1 + 1 + 1 + 1 + 1 + 1 + 1, then 5 is written 1 + 1 + 1 + 1 + 1; and the symbols are a good expression of the definition of these numbers as we have seen already. In the same way their addition is well expressed by the symbol: (1 + 1 + 1 + 1 + 1 + 1 + 1) + (1 + 1 + 1 + 1 + 1); this will be the subject of our judgment; on the other the predicate will be: 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
As Mercier says, it is sufficient to remove the brackets for the identity of the subject and the predicate to be made clear in a sense-perceptible way.

But the mere removal of the brackets does not seem to be sufficient; for it is also required, and that with absolute necessity, that we see whether the so-called bi-univocal correspondence between the elements of the subject (after the removal of the brackets) and the predicate is really present; if this is present we are sure of the identity. But this is the same experiemental element that we have found above. The only difference is this: we were looking for the number (as yet unknown) which results from addition, and to find this we successively added 5 units. Mercier supposes this number to be 12 and by comparing this with the result of the addition, effected in a single act, he verifies his supposition; but a similar computation or comparison of the individual elements of the predicate with those of the subject is involved here. And here, as above, this operation is experimental, but it is an experiment which in the matter in question implies an intuition of the nexus. A purely analytic argument is not sufficient. Add the fact that we must know that the removal of the brackets (i.e. the operation of addition itself) leads to a necessary effect; and this again implies an experiment which produces an intuition.

9. The Theory of Cajetan. We will now draw the doctrinal conclusion from these examples. It seems that such an experimental element not only can but must enter into the consideration of our mind; not only so that we may gain the notions, but also so that we may see the necessary nexus between these notions, that we may make a judgment. The theory of Cajetan (which is the theory of S. Thomas, as we have proved in our first article) seems de facto to be verified. He expresses it thus:

"over and above the concept of the incomplex terms, it is necessary to posit some determinant or motive of the intellect leading to the making of such a composition. Such a moving-principle must be the sense, for before the knowledge of the first principles, Aristotle knows of no moving-principle of the intellect except the sense. Hence the complex cognition of the principles necessarily pre-requires experimental sense-knowledge - - -. Therefore we must admit that sense-experiment is pre-required for the generation of the habit of the principles (by reason of the complex knowledge) because it is the proper moving-principle or determinant of the intellect to this end, and because it is the means essentially ordered to this."

Aristotle had already enuntiated this doctrine in the Prior Analytics: "For this reason the proper principles of each ssience must be transmitted to (put down to) experience".

And so we see the broad opposition there is between the theory of Aristotle and those of Plato, Kant and Stuart Mill. The three latter agree in this, that experience can never produce necessary judgements. Hence Plato's appeal to the memory of a prior intuition of the pure forms; hence the rationalism of Kant; to save the necessity of the principles of mathematics he 'invents' the subjective a priori forms of our sensibility; hence positivistic empiricism denied the true necessity of these judgments. But each of these three theories presupposes, and indeed without examination or argument, that experience cannot generate necessary judgments and all three are wrong in this supposition as is clear from our exposition above. We have done no more than pay attention to the clear data of our internal experience which attests both the experimental origin (from the sense-data) and the necessity of these judgments.

True, not every experience is of this nature, but in the examples studied it is certainly found. If, on account of this influence of experience, anyone chooses to call these judgments synthetic, he will have, certainly, necessary synthetic judgments, but certainly not synthetic a priori judgment in the Kantian sense; and moreover <u>because</u> Kant supposed that such judgements could not be derived from experience, he thought up his theory for that very reason. But this supposition of his was gratuitous and, as our exposition shows, it is false; hence <u>these</u> necessary synthetic judgments undermine the very foundation of his doctrine.

Later we will see how what we have discovered by attending to the data of our experience can be theoretically explained. Here it will suffice to recall what we have already adumbrated in our previous article: as the notions (universals) which represent the nature of things to us can be derived from experience by the abstraction of the agent intellect, so there can be (and in fact there sometimes are) cases in which the nexus also, i.e. the nature of the nexus (from which necessity and universality follow) becomes clear to us by the abstraction of the agent intellect alone. It was proved in that article that this is the theory of Aristotle and S. Thomas; we can briefly sum up the theory in the words of Cajetan quoted there (pg 158):

"For not only the universal concepts of the terms must be generated, but also their complexion (i.e. nexus), and this is related to experimental complexion in the same way as the knowledge of the terms is to their repeated sense-apprehension".

Thus this abstraction of the nature of the nexus is present in one case but not in the other; we will later examine the conditions for its presence.

10\. - Judgmentsx which must precede. Abvoe we spoke of the subject and predicate of this proposition 7 + 5 = 12. Not infrequently philosophers and mathematicians have great difficulty in explaining the structure of this judgment, i.e. in describing what precisely is its subject and its predicate; nor are they always successful (See E. Meyerson in his book 'Du cheminement de la pensée' II pg. 333-339, III 877-879).

We saw that a good explanation was given by Mercier, who thus describes the subject of this proposition: it is the sum (la somme) of two numbers (determined numbers) 7 and 5, i.e. the number which results from the addition of those numbers (its specific determination is unknown to us before the judgment); and of this resultant number it is enuntiated that it is the determinate number 12. The judgment can therefore be enuntiated thus: the number which results from the addition of 5 and 7 is the number 12.

It is clear that a twofold necessary, intuitive knowledge is presupposed here. One is the following: it is presupposed that the addition of a number to a number (or more universally of a multiplicity to a multiplicity) generates another number (multiplicity) - and in fact a greater number; and that in such a way that <u>necessarily</u> (hence always) from the addition of a specifically determined multiplicity to another so determined there results in turn a third determined multiplicity (specifically one and greater). This appears even more clearly in the Greek expression, where for (the) number which results one says 'the number' (   ), or in Italian which says 'il numero'.

And this precedent knowledge is again only attained by an experimental intuition, but with such facility that it is not expressed. We found similar instances in the analysis of the systems of Leibnitz and Mercier.

But there is another knowledge which is antecedent to the one just examined; and it is the following: a number <u>can be added</u> to a number; in other words: addition (possible) is the <u>proper passion</u> of a number; or more universally: quantities of the same kind can be added together. And this indeed seems to be the 'first known' in this field (of mathematics). If we examine the source of this necessary knowledge we arrive at only one conclusion: from inspection of the phantasm; and this again points to the experimental element; we find the saying of Aristotle verified once again: "must be transmitted to experience" (put down to experience); and the dictum of Cajetan: the nexus of the terms "is related to the experimental complexion, as the knowledge of the terms is to their repeated apprehension"; it is known by intuitive abstraction.

We find that this is so often, if not always. In judgments which are <u>enuntiated</u> as principles, there are concealed antecedent cognitions of a certain primary proper passion of the subject which are not themselves enuntiated because they are so clearly grasped by such a simple intuition, and are easily omitted in the "resolution" of the propositions to first principles. We will treat of this more fully later.

Our judgment "7 + 5 = 12" with its presuppositions includes therefore the following judgments: a number can necessarily be added to a number; from the addition, again necessarily, there results a number; from specifically determined numbers a specifically determined number results; hence necessarily a number and this itself is specifically determined. What is it? The answer cannot be found except by some experimental computation (counting); but this leads to our judgment as a necessary judgment.

§ 2. On the Necessity of Immediate Judgments in Geometry.

1. The twofold problem in Geometrical Knowledge.

As in Section 1 we discovered certain immediate (because drawn from an experience) and necessary arithmetical judgments, in the same way such judgments are to be found in geometrical knowledge. But there is a vast difference between the two sciences. For the sense-data from which we abstract what we affirm in the judgment - both terms and their nexus - are exact in the case of arithmetical science: here the numbers with which we are dealing are well determined even in sense-knowledge, for they consist of elements, individual entities, which are clearly distinct and separate. Hence in those elementary arithmetical judgments the problem of exactitude does not arise. For in sense-knowledge - we do not say in the real order - there are no points without any extension, for these cannot be perceived by the sense; nor are there lines without width; nor indeed, as it seems to us, are there surfaces without depth; here we cannot determine what is the precise point which divides a line into perfectly equal parts, we cannot determine exactly whether a line is really straight, whether lines meet exactly in one point.

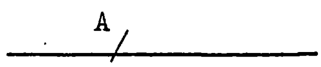
Now we will only deal with the first problem, we will consider the second in the following chapter; the outcome of our exposition will show that these two problems can not only be distinguished but also separated. Therefore when in this chapter we speak of points, these are to be understood as they are found in sense-data, i.e. as small corpuscles, nor need they be the very smallest visible objects; thus 'a line will be a surface whose width is very small; a 'straight line' will be one which appears to the senses as such, and so on. Concerning these objects we can discover immediate judgments derived from experience and bearing on the nexus between the subject and the predicate, and these judgments are nonetheless necessary.

2 - Extension and "Space". The object of geometry is extended being as extended. It is often said that "space" is the object of geometry; if this word is understood as a synonym for "extended being as extended", there is no error. But the word "space" is also used for "absolute" or "imaginary" space i.e. for a receptacle which contains in itself all extended bodies and precedes them, which remains when all bodies have been destroyed, which ultimately is only an ens rationis (a pure mental construct). If we use the word "space" there is a danger that we may slip from one meaning of the word to the other, and so fall into error. Further the notion of "extension" is prior to that of "space"; for this is conceived as an "extended" receptacle; and hence only because and inasmuch, as it is extended, is it the object of geometry. The notion of space is only introduced after the consideration of the mutual local relations of bodies, i.e. of real extended entities. If anyone considers four of the five arguments with which, in his Aesthetics, Kant tries to prove that "space" is a form of the subjects sensibility, in such a way that he replaces "space" conceived as the receptacle of bodies, with the notion of 'that which is extended', he will see immediately that those arguments do not hold. Extension itself cannot be made a mere form of the subject.

Moreover in purely geometrical considerations we speak of the movement of figures, and this we will later justify. But these figures cannot be parts of space, because these cannot move; these figures are real extended entities whose movement can rightly be considered.

3 - On Divisibility. We will consider extended being; in the following chapter we will see more accurately that extension as it is known by us by abstraction from bodies has three dimensions; we will see also how we arrive at the exact notions of surface, line, point. Now let us take as our example any inexact "line", as we perceive it in sense and imagination. It is a type of extended being in which this notion is perfectly verified.

We grasp immediately that there can be a longer and a shorter line, we affirm this judgment "there is greater and less", and in this we express what is rightly described by Aristotle as the "specific property of quantity".

How do we understand this truth? In a drawing or in imagination we divide a line by a point A: 

and "in this phantasm" we see that there are lesser lines, the parts into which this line can be divided. Let us compare this experience with a physical experiment regarding the same line. Let the line be white or black, and the point A be red. In the same way we see that the point A divides the line into parts. There is however a vast difference between this experience and the former one. We can express the first by saying: this extension is divided by the point A; the second by: this white is divided by this red. Is it necessary that it (the white) be extended? Certainly. Thus, we intuit in the sense-data, in the phantasm, that the possibility of dividion necessarily flows from the fact the line is extended and that it is only contingently combined with the whiteness or blackness (of the line). Therefore we must express what we experience more accurately. In the first case that must be done by a causal (or rational) proposition:

this, because it is extended, is divisible (or: this is extended, therefore it is divisible).

Note, however, that (in spite of the words "because" and "therefore") we have not here a truncated argument; there is a three-fold experience in our mind: this is extended and this is divisible, and thirdly: we experience (in our mind) the necessity of the nexus between the first and the second. In the second case what we experience must be expressed by a purely copulative proposition: this is white and it is divisible. It is a purely accidental, contingent judgment.

Our experience in a mathematical case must be expressed by a causal proposition: this (concrete object) is extended and therefore it is divisible. In a concrete case we already understand the necessity, we understand that divisibility is a consequence of being extended. Returning to this singular causal proposition we get: this because it is extended is divisible. Nor is this a result of a reasoning process, it also is got by intuition and so we discover the 'specific property of quantity' according to Aristotle. So in the case of the judgment "7 + 5 = 12" the first precedent cognition was the knowledge of the 'specific property' of numbers.

The second proposition can be examined in a similar way: a straight line can be prolonged; this again is a specific property resulting from the fact that it is an extended entity (and this we know immediately) only accidentally linked to the fact that it is white.

4. - On the Order between Points. Let us take an example where there is question of a judgment regarding the order between points. In the work of Hilbert "axioms of order" make up an integral series. In a "straight line" let there be three "points" in this order: A, B, C. We determine a fourth point so that it can be defined: a point on the line between A and B. The question is asked: is it also between A anc C? When we have looked at the diagram (figure) the immediate answer is: Yes and

necessarily so. Then it is asked: is it also between B anc C? After inspection of the diagram the answer is: this is impossible. Let us compare this with a case in which D is defined by a physical character. D is a red point on the line. By looking at the diagram we can answer the same questions; and the answer will be: in one case it is between A and C, in another it is not; or it is between B anc C, but in another case it is not. We make our judgment after the same experiment: we look at the diagram. But in a 'mathematical' case besides the grasp of the nexus between the subject and the predicate of the judgment, we have an intuition of the necessity of this nexus; in a "physical" case of the contingency of the nexus. That the predicate is affirmed of the subject (or in negative judgments denied) we gather from experience: that in the first case the nexus is necessary we also grasp intuitively in the concrete instance, but by the intellect's intuition of the phantasm, which in that case penetrates to the essence. In the words of Aristotle quoted above: "Thus the intellect understands the forms in the phantasms".

NOTE: One of these judgments was about an impossibility: it is impossible for D to be between B and C. Let us consider this case for a moment. We not only make the negative judgment i.e. we do not see the possibility of this position of the point D (defined "between A and B"); but we make the positive judgment: we see the impossibility of this position. This observation is important; let us compare this example with another, with the question: whether an extension of four dimensions is possible. If we appeal to our imagination in the same way we will get this result: first, we understand the possibility of an extension of three dimensions; then, if we ask about a fourth, what the phantasm tells us can only be: we do not see its possibility, but from this we, have not yet got: the possibility of a fourth dimension is positively excluded as a result of this. It is worth while to attend carefully to each case and the difference between them in our internal experience; in one case (that of point D) the impossibility is clearly seen to arise from the data, in the other case this necessary resultancy from the data is certainly not present.

5. - <u>On Pasch's Axiom</u>. Let us consider a 'plane' figure, a triangle. We take a truth which is similar to the geometric axiom which was first enuntiated by Pasch (in 1882!) We draw a triangle i.e. a figure consisting of three "straight lines" which intersect in three points. We draw another straight line through the angle of the vertex. We make the judgment: if this line is sufficiently prolonged it will necessarily cut the opposite side, the base of the triangle. And once again we intellectually intuit the necessity of this result in the sense-data. Once again let us make the comparison with a physical case. If the line which is drawn through the vertex is then defined, not as that "which is drawn through the vertical angle", but "which is red", it would not then follow that it would cut the base. If peraccidens in fact it does fall within that angle, it will cut it, if it does not, it will not. And in those cases where it does, it does so not because it is red, but because it has such a position, as we have already explained above in the case of the points.

The inverse proposition also is true in the 'mathematical' case. The line which joins a point in the base to the vertex falls within the vertical angle. In the first proposition the subject was defined thus: "a straight line within the angle" and from that determination the predicate results with necessity; "it cuts the base". In the second proposition the matter is reversed but here also the nexus is known as necessary. In each case one results from the other. And in each case we need a datum of sense to know <u>what is the predicate</u>, which, <u>as we already intuit in advance</u>, will be necessarily connected with the subject. This last point will be made clearer in the examples which follow.

6\. - <u>On the Sections of a Cylinder and 'Moebius' strip'</u>. There are other examples from 'topology' (which in our case of "inexactitude" can also be called morphology). Let us imagine, or let us actually take, some sheets of any material which have an almost cylindrical shape. We can cut them, even physically in two ways: either (roughly) along the line which goes from the top to the bottom or perpendicularly to the axis of the cylinder. What is the result in the first case? As is apparent from looking at the phantasm, or from the physical execution of the division: there results one surface which is still coherent (which can be unfolded flat). What results in the second case? There result, as is clear in the same manner, two cylindrical surfaces no longer coherent. We see the experimental element. However it is not a pure physical experiment; for again we have a case where we intuit the nexus between the subject (the cylindrical surface which is divided in such and such a way) and the predicate (it is divided into one or two surfaces respectively). We intuit that the predicate necessarily results from the operation performed on the object. (We will presently examine the case more closely). One who has a sufficiently vivid imagination will be able to see this in the phantasm: he who has not, will need the physical performance of the operation, but even here, before he sees the result, he intuits that the effect which follows necessarily results from the operation; in order to know <u>what</u> the effect is he needs a physical experiment (but one is sufficient); this will be even clearer in the example which follows.

<u>'Moebius' Strip</u>! There is question of a surface so named because it was discovered by Moebius, a mathematician of the last century. We can construct it physically in this way. We take a sheet (v.g. of paper) with the shape of an oblong rectangle of small width; its extremities (the small sides), after being twisted in a half-revolution (180°) are glued together. Thus we get a surface, which is like a cylinder (rather low), but in it there is one twist; the operation is easily performed. This surface is 'Moebius' Strip'. Later we will consider it in another question.

We now cut this sheet along a line drawn in the middle of the sheet parallel to its edges; we have now a division which corresponds to the second 'cutting' which we performed above on the cylindrical sheet and from which two cylinders resulted. What will be the result of this cutting? Many will not have a sufficiently vivid imagination to be able to 'read' the result in the phantasm. But the actual execution of the division shows that two surfaces do not result but only one, which is like the first but has two twists. (If this sheet is again subjected to the same operation there will result two surfaces, but which are joined together like two links of a chain. This certainly cannot be grasped except after the realization of the operation.

So, in order to affirm this effect in our judgment we need sense-experience. But this experience is associated with the intuition which we have of the necessity of that which we thus learn from experience. In this it differs from an ordinary physical experience: but it is the same in this, that before the experiment we do not possess the notion of the predicate which is to be affirmed of the subject. We know certainly: from that cutting of the plane figure some plane surface would result; the genus was known but in order that the effect should be determined in our knowledge according to its specific essence, we required experience, and indeed real experience, but one experiment is sufficient.

It should cause no surprise that here we have an intuition of necessity both with respect to genus and to specific difference. Before the experiment we knew that the effect of the operation does not depend on the material of the surface; it does not matter whether it is metal or wood, or paper, or elastic. The properties of the material may have an influence on the force we will have to use in the cutting, but not on the topological structure. The only factor that is relevant to this is the shape of the object inasmuch as it is extended. We do not derive this independance of the nature of the material from this last experience, we we intuit it by abstraction (about which we will speak presently) and therefore this experiment becomes a truly intellectual experience.

In the genesis of this necessary judgment the experience itself really plays its part as a motive; it determines what is to be affirmed, it determines the specification of the predicate, it does not determine the necessity. The saying of Cajetan cited above is verified to the letter:

"Over and above the concept of the terms one must posit something which determines or moves the intellect to make such a composition. This motive must be the sense... For the generation of the habit of the principles experience is a pre-requisite (by reason of the complex cognition) since it is the proper motive or determinant of the intellect to this end and since it is the means essentially ordained to this."

§ 3. The Doctrine which is derived from these examples

1. Comparison with Physical Laws. In each of these examples we intuitively grasp the necessity of the nexus between the subject and the predicate of the judgment by which we affirm: it is necessarily so. Nevertheless in all of them (each in its own way) there is the influence of the experimental element, so that the judgment is of the kind which in modern languages is called 'a finding' ('verification'); but it is an 'intellectual finding' just as it is an intellectual intuition, linked certainly to abstraction from the senses. Goblot had already noted something similar, but only in reasoning processes, and he wrongly calls it 'logical confirmation'. This experimental element, in more simple cases, can be what is today called 'mental experiment' i.e. carried out by means of the phantasm alone, without the actual perception of the exterior senses. But in more complicated cases, in the division of larger numbers (as in the first example), in the computation of prime numbers, even in the addition of 5 and 7, in the division and subdivision of the strip Moebius, most certainly in the

planes of Stahl, we must also use the experience of the external senses, even the assistance of a physical operation.

Let us compare this again with a physical judgment, even a certain one, which is expressed in a physical law. "If a glass rod is rubbed with a silk cloth, it attracts light bodies, v.g. pieces of paper". Certainly we do not immediately enuntiate this judgment as general and certain, but only after many experiments, by physical induction (in the modern sense). Another example could be: "a heavy body which is released here falls to the ground". Certainly no one would doubt about either judgment. Do they express a necessity of nature? We can reach certainty on this question on the evidence of the regularity of the facts. But do we intuit the necessity, is the nature of the nexus between the subject and the predicate ("glass rubbed exerts attraction") "a heavy body falls") clear to us? If we compare these physical laws with each of the examples given above we observe a very great difference. In each of the mathematical cases we understand that the predicate is necessarily connected with the subject, flows from it; even in cases in which only after the experiment, after the operation, we know what precisely the predicate is. Even the understanding of this difference between the mathematical and physical cases is a kind of "finding" of internal intellectual experience; in the mathematical cases we not only have an intuition of the subject and the predicate but also of the necessary nature of the nexus, this intuition is lacking in the physical cases. And so in physical judgments the intellect only repeats (in a general way if the case permits) what the sense perceives in the others the intellect transcends the senses. It will be profitable to devote more attention to this difference; we will presently return to thosephysical examples.

2. - More on the Theories of Geometrical Knowledge in relation to Experience.

If anyone fully grasps the significance of this difference he will immediately understand the falseness of the empiricst position of Hume and Stuart Mill. They think that in all these cases, mathematical as well as physical, one has the impression of the necessity of the nexus only because he has arrived at it by dint of long-established habit; hence he expects the same in the future. Even if this position were true in physical cases (which can in nowise be conceded for all such cases) it is immediately clear that it is false in mathematical cases. Here there is evidently more than expectation arising from custom; here there is an intuition of the necessity of the nexus; this is clear to us from internal experience itself; the empiricist position simply falsifies the data of this internal experience.

To put it briefly: Empiricism neglects the difference between physical and mathematical cases which consists in this, that in physical knowledge the intuition of the nature of the nexus is lacking whereas in geometry it is clearly present.

But also every theory which assumes as its foundation: "from experience a necessary judgment cannot be derived", every rationalist theory therefore, is devoid of value from the start. So it is with the theory of Plato, which besides no one defends any longer. It is the same with the theory of Kant, who actually makes this proposition into the principle and foundation of his doctrine. He makes this assertion without any justification; thus there can be no question of examining his arguments on this matter; he simply makes the assertion, neglecting the experimental element in geometry. This can only be explained by the fact that, in advancing this proposition, he only attends to physical laws, or because he accepted it on the authority of Hume. That this principle of Kant is false follows from our whole exposition, or rather from the individual examples which we have seen.

Therefore the doctrine of Aristotle must be accepted: there is nothing in the intellect which was not first in the sense; and further this doctrine is not only to be understood of the notions themselves considered separately but, with Airstotle, also of the existence of the nexus between the subject and the predicate which is affirmed in the judgment; the consideration of the notions of the subject and the predicate is not sufficient; to affirm the nexus we once again (or at the same time) need experience. This is the doctrine of Aristotle in the last chapter of the Posterior Analytics, where he treats of the origin of the first principles of science; this we have already heard in two texts which we repeat here: Anal. Prio. I. 30, 46a 17: 'Thus the principles of each science must be made over to (put down to) experience' and De An. III 7, 431 b 2: 'The intellect understands the forms in the phantasms'.

This is the doctrine S.Thomas holds when he attributes the origin of the principles, as well as the notions, to the abstraction of the agent intellect, as we have proved in our first article. This doctrine he also puts forward in his commentary on the Posterior Analytics in the last lectio. It will help to quote him here, because what he says brings us back again to a useful comparison. In I Sent. Dist. III q 1, a 2 he states:

"Things which are per se known to us are made known straightaway by the sense; as when we see a whole and a part we know straightaway that every whole is greater than its part without any enquiry".

Let us compare this example with a similar 'physical' one. We see two surfaces; a larger one which is red, a smaller which is blue; and in three judgments we express what we see: A is red and B is blue and A is greater than B. It is a purely copulative judgment. Then we look at two surfaces of which A is the whole and B is a part and we judge: A is the whole and B is its part; <u>therefore</u> A is greater than B. We have a causal proposition; on account of this intuition which is lacking in the first case. The relation of size results necessarily from the relation of a whole and a part.

This doctrine seems to be abandoned by those modern scholastics who think that the consideration of the notions is sufficient without the experience of the nexus; but quite clearly it is still maintained and defended by Cajetan. We have only to recall his words quoted in the last paragraph above.

To understand better how sense-data are "proper <u>motive</u> or <u>determinant</u> of the intellect" in affirming a principle, let us attend again to another example already given - that of the relation "between" (things), where to three points A, B, C, situated in this order on a line, we added a fourth D, between A and B. From the position of point D (between A and B) we understand that there results necessarily a relationship of position both with respect to A and C, and with respect to B and C; and then we express this relation by stating the predicate. This on the evidence of our consciousness, cannot be done without looking at the figure (diagram) either in imagination or actually drawn.

How do we de facto affirm: "it is necessarily so". Certainly from the operation of the intellect abstracting and intuiting. How do we arrive at <u>what</u> is affirmed (namely such a position in relation to A and C or in relation to B and C)? From sense-intuition. In this respect, and in this respect only, the words of Cajetan are true: the sense-datum is "the motive or determinant".

3. - <u>Implicit Judgments</u>. This determination which Cajetan demands is not always realized in the same way nor does it always impel us to an <u>explicit judgment</u>. We now wish to pursue one point.

The judgments which are explicitly affirmed, and even more, those which are enuntiated, are not usually the most elementary judgments. We can verify that from almost every one of the examples examined above. Here are some of them.

In the first case we enquired how many are the divisions of the number 5040. The final operation was the counting of the divisions we had found. To begin with there was the series of equal divisions of this number: into ones, twos and so on. But this operation presupposes this judgment: this number is divisible into units and smaller numbers. This divisibility is in fact (besides the possibility of addition) the first property of number; that it is divisible into smaller numbers and units is a property which follows on this first property; that these lesser numbers can be equal (that is not always the case as Plato observed) and how many they are, is again a property which only then follows and can be known. But we usually only attend to the final judgment and we do not usually enuntiate the judgments which affirm the two prior properties. In this respect they are, and frequently remain, merely implicit.

We find the same in the question of the number of prime numbers, of combinations, of the addition of 5 and 7. In geometrical cases it is the same. The first property of extension is: it can be divided; the second is: by division we again get extensions of the same kind, from a line a line and so on. And vice versa two extensions of the same kind can be added and a greater extension of the same kind results. A cylindrical surface or Moebius' slip can be divided; again from this division there result surfaces. How many and what kind they are comes to light by an experiment either mental or real. Even in these cases the judgments relating to those first properties, although they are necessarily presupposed, are not explicitly made by the person who makes the final judgment (on the result of the experiment). The movement of the mind (le mouvement de la pensée) implies such judgments, but the single stages of this movement are not enuntiated.

A fortiori this movement is not explicitly syllogistic; in fact not even implicitly so. To return to the addition 5 + 7 = 12 once again. To make sense of this judgment - as we showed above - the following truths must be presupposed: a number can be added to a number; next: the addition produces a (greater) number; then only can this number be specifically determined by actual counting. This movement does not proceed syllogistically; if that were the case it would proceed something like this:

1) Every number can be added to another number. But 7 and 5 are numbers. Therefore 7 and 5 can be added together.

2) Every addition of numbers produces a number.
But 7 and 5 are numbers.
Therefore their addition produces a number.

3) Every number resulting from an addition can be specified by counting. But (as is clear from experiment) in specifically determining the number which results from the addition of 5 to 7, we get 12 by counting. Therefore 7 + 5 = 12.

It is clear that our mental process is different. We do not need to know these three universal majors before this process. Considering these numbers (7 and 5) we immediately detect that they can be added together, and that necessarily, in a particular case we understand by "inspecting the phantasm" the necessity of this property; and that is sufficient. Again in the same way we understand, in this particular case, that from their addition there results necessarily a (greater) number. Thus the process is not the following: We know the universal truth (the majors) and we discover (in the minors) that they can be applied here; but rather this: we discover immediately the necessary truth of the two conclusions of those syllogisms, one straight after the other: (and if we don't wish to go on to count the resulting number we can go back to these truths and say: this number and that second number, inasmuch as they are numbers - for we know this by intuition - can be added together. Therefore every number can be added to any other number. But here we have a movement opposite to that syllogistic movement. Then by employing counting (again without a syllogism) we find the number 12. That process is in no way syllogistic, not even implicitly; the syllogisms are not only not expressed but, as is apparent from our analysis, they are not present in our mind. Nonetheless there is in our mind a progressive process from ohe to the other.

4. - Virtual Judgments Thus those judgments relating to the primary properties and not enuntiated and in this sense they are implicit. Perhaps we should

say more: those judgments are not affirmed mentally (except in analysis such as we have presented here), although they are certainly virtually present.

This is our understanding of the matter. We are using the well-known theory of S.Thomas according to which the judgment consists in reflexion on the first operation of our mind (that is on the conception composed of two connected terms); in this reflexion our mind knows the nature of this act (the first operation) and so it affirms i.e. says: it is so, it judges. We will say more on this theory later.

Now, in the case of the judgment we are considering (7 + 5 = 12) the whole process can be explained in this way. Considering these numbers together in its first operation, our mind (by attending to the phantasm) already has the idea of these numbers together with the nexus of this idea with its property i.e. with the possibility of addition, indeed the "necessity of the nexus" is immediately clear to the intellect but without stopping to affirm, "it is so", (this is necessary for a genuine judgment) it proceeds in its movement (it is still always 'the first operation of the mind') and it discovers always in the same phantasm the addition and its connexion with the result (the greater number), it discovers at the same time (immediately), by attending to the same phantasm, the "necessity" of this nexus; once again without stopping to affirm this nexus it begins to count this number.

It is in this way that the process which leads to the judgment 7 + 5 = 12 is to be explained; but obviously it is performed more quickly than it has taken us to describe it. If this is the case, not only does the process not consist of a series of syllogisms, neither does it consist of implicit judgments i.e. which are affirmed but not enuntiated, which are mental affirmations; these judgments are only virtually present and at each stage can become actual. On this theory we can easily explain the fact which commonly arises in "analytics" i.e. in resolution to first principles. It is often difficult to explicitly enuntiate all the first principles. Sometimes, even after long and laborious analysis, there remain principles which are always applied but never enuntiated, even by those who intended to enuntiate them all. So in geometry, the axioms of order were never enuntiated before Pasch (1882), although always applied; they remain 'latent' because they are so clear, so '   '. And in theory that is all the more easily explained because in the mind itself they are only present as elements of the first operation of the mind which, before they are affirmed, are immediately combined with new ideas into a more complex idea; these affirmations are only virtually present in the judgment which follows. And the two truths which are presupposed before the judgment 7 + 5 = 12, seem to be of this kind; for we must admit we do not know where they are enuntiated. And perhaps for the same reason many of the points we have made in this chapter, not about mathematical principles, but concerning the mode of operation of our mind, are not usually explicitly stated. But it seems clear that these considerations are necessary in a general philosophy of human knowledge.

(9) 4. On the Material and Formal Nexus

1. Thus the theory of Aristotle seems to be the only one which takes account of what reflexion on the origin of these mathematical judgments teaches us; it is the only one which does not deny to experience the influence which it in fact has. But how do we explain this if, on the other hand, in physical judgments, which have the same origin, this intuition of the necessity of the nexus is lacking, although it can be present. We will offer a full explanation later; here we will add some clarifications which we hope will be sufficient.

2. - The material and formal nexus

In every judgment we affirm (or deny, if the judgment is negative) the nexus between the subject and predicate. At least this nexus is present, that the form which defines or designates the subject (if it is not a question of those judgments in which the subject only indicates the supposition:

'this' is white) is found in the same supposition in which the predicate is found, the latter being predicated of the same supposition in the judgment. "Extension is divisible"; the form "extension" defines the subject, the form of "divisibility" determines the predicate, and both forms qualify (inform) the same supposition. There is "material identity", as it is often called, between the subject and the predicate, because their forms inform the same "matter". S.Thomas is concerned with this identity when he says: "The intellect expressed the unity of a thing by a composition of words, which is the note (sign) of identity" (S.contra Gent. I,36).

This <u>nexus</u> itself can be called <u>material</u>; and it will be <u>purely</u> material if there is question of a judgment which is contingent: "this ball is white". In this judgment there is no other nexus between the roundness and the whiteness than the fact that these two forms merely per accidens inform the same suppositum.

But there can be a closer nexus between two such forms which determine a suppositum as the subject and as the predicate, namely the nexus which results from the very nature of the forms.

These are cases in which the form of the predicate results from the form of the subject itself; v.g. "Extension is divisible", "number is divisible, "this specific number (5040) can be divided into twos, threes etc." In such a case in any suppositum which is informed by the form of the subject, the form of the predicate will <u>necessarily</u> be present. This is especially verified in "the fourth mode of predication per se", "according to which this preposition '<u>per</u>'designates the relation of the efficient cause or of any other "(S.Thomas Analyt. Poster. I lect. 10,n.7)

There are other cases in which the form of the subject necessarily <u>presupposes</u> the form of the predicate v.g. "What acts exists". " I who think, am".

In all these examples there is not ohly a material nexus (purely material) but a nexus which can be called <u>formal</u>. The predicate must be affirmed of the subject "by reason of the form implied by the subject"; thus S.Thomas (III Sent. Dist. 11 2.1 a.4 ad 6): "For the truth of a proposition it is sufficient that the predicate be applicable to the subject in any way; but that a proposition be 'per se', it is necessary that it be applicable to it by reason of the form implied by the subject."

3. - Physical Laws and Mathematical Principles.

In cases where there is a formal nexus, a distinction must be made (and this may need to be subdistinguished): either it is clear to us that the nexus is formal, or this may be more or less obscure to us, although the formal nexus is de facto present. In the second case the judgment will be about a nexus which is necessary, "in necessary matter", but the necessity of which is not known to us. Hence this judgment differs completely from that enuntiated above: "this ball is white"; in judgment we know that the nexus is purely material. To verify this it is simply enough for us to have once seen a ball which is not white.

The second case which we have described often occurs in the physical sciences, which from long and often repeated experiments discover general physical laws which, as experience teaches, are always verified. Let us consider the examples already given: " a glass rod rubbed with a silk cloth attracts small bodies"; "this body left to itself falls to the ground". Hence we at least suspect, or rather are convinced, that here also there is a necessary nexus between the subject and the predicate, here also between both forms, that of the subject and of the predicate, there is not only a material but a formal nexus (or also both forms have a formal nexus with a third which is unknown to us).

This means that we can now further explaih the difference which we found de facto between physical and mathematical cases. The greatest difference is not this: that in physical cases only after lengthy and repeated experiments can a general law be verified (in experiments where there is question of determining specific "constants", one experiment can be sufficient), whereas in mathematics one experiment, and even frequently a "mental experiment" performed in the imagination, is sufficient. But for one experiment to be sufficient, even in mathematics, it is a pre-requisite that there be a lengthy experience and preparation in the sense-faculties, which begins in infancy, so that the notions be made reasonably clear to us and can be abstracted. That is also the doctrine of Aristotle and S.Thomas at the end of the Posterior Analytics, of Cajetan also, who gives a good description of this process of preparation of the senses (see the quotation in our article 'On the Origin of the first Principles' pg. 159). But the greatest difference lies in this: in mathematics we intuit the necessity, in physical cases we conclude to it. Thus it is that in physical cases, even after lengthy experiment, we only see the material nexus (certainly constant) not the formal nexus (which should be present); in mathematics we see both the material nexus (both forms are in the same suppositum) and the formal nexus (by intellectual intuition of course).

We will now make a further step in our explanation. First we note one point in passing. Those general physical judgments suppose, as we have said, the necessity of the nexus; hence they suppose the formal nexus between the subject and the predicate; hence they should be intelligible in se (in themselves). But this intelligibility is not clear to us in a purely physical law. Because, nevertheless, it should be present, it is the further task of science to bring it to light. And precisely this end is served by physical theories which try to give to laws this intelligibility, the formal nexus itself. But an examination of the way in which science and natural philosophy attain this end is not the proper subject of these lectures. Much will be found on this question in our Cosmology.

4. - On Formal Abstraction. From our internal experience we know: we have judgments (mathematical cases) in which we intuit the formal nexus between the subject and the predicate from the sole inspection of the sense-data ("to inspect the phantasms"); there are other judgments of which we

are certain (the case of a true physical law) but in which we do not see the formal nexus by the sole inspection of the sense-data. Let us consider the reason for this.

How can we formulate a well-determined physical law in such a way that the formal nexus, as yet not clear to us, may be made evident to us? By means of physical induction, as it is called, in which the circumstances of the experiments are subjected to variations so that those properties of the bodies involved which have no influence on the outcome of the experiments (i.e. on the predicate to be determined) are excluded from the enuntiation, and the others retained. This is done according to the methods which were more or less well described by Stuart Mill. In this way we are assisted in forming our judgment as to what properties are required in a body so that a certain consequence should follow. These properties constitute a form, which determines the subject in the judgment expressing the law, the consequence which will follow will be the predicate. This method can be called "physical abstraction" or "experimental abstraction"; for by selecting different circumstances we leave out now one now another property which can have an effect on the outcome; we therefore "abstract physically" from them. By submitting the object in question to this method we can finally reach the judgment which expresses the formal nexus, although its nature is not evident to us.

So as we have said above, we are left with the necessary task of physical theory which must enquire into the nature of this nexus; we know that there is an intelligibility present; what it is we do not yet know.

The case is very different in mathematics. Recall the example of the division of cylindrical sheets; from a transverse section the result is two sheets again cylindrical in shape, from a longitudinal section the result is one surface which can be made flat. Do these results depend on the colour, hardness and the other innumerable qualities of the material of the cylinders? Certainly not. Is this clear, as above in the physical law, by repeated experiments which select and examine the matter of different qualities? Certainly not. Each of us immediately understood the result, either in one operation which was either physical or, since probably none of us has ever divided a cylindrical sheet in this way, purely in the imagination. So we see: our mind itself has the capacity in such a case to abstract the form which is significant from the matter and the forms which are not significant. The form which is significant here is the extension with its shape and we understand immediately that it depends on this factor alone what will result from a definite section (what this is the experiment will subsequently show); we understand immediately that this does not depend on the colours, on the other qualities, on the physical essence of the body which is being divided. It is in this positive understanding that the abstraction consists, not only in not considering the sensible qualities.

Thus we get what is well called formal abstraction by the scholastics, i.e. the abstraction of the form (clear to us) from the matter (at least not so clear to us).

This abstraction is found where our mind can abstract something (a form) in such a way that we intuitively grasp that some property necessarily results from it, i.e. is linked to it by a formal nexus. And it is the function of internal experience to detect such abstractions.

But in establishing a physical law we need many and varied experiments to establish the "form of the subject" which is significant (i.e. to which alone the form of the predicate is invariably linked); this predicate is known from experience, from sense data. In mathematical cases instead of the former method we have formal abstraction, and hence we intuit the formal nexus, hence what is predicated of the subject will be necessary. But for us to know what is to be predicated we must attend to the sense data, either in imagination, or with the help of the external senses, which is (in the words of Cajetan) "the proper motive and determinant" of the predicate.

Even in physical cases we can approach this perfection, so that from one experiment we can attain to the knowledge of a specifically universal law. We now know that pure bodies have specifically determined "constants"; v.g. they have a constant point of fusion or boiling-point. If we have prepared a new body we determine its point of fusion by one experiment; and this knowledge is an universal law in this specific matter. The knowledge of the nature of the nexus, the formal nexus, is however lacking.

5. - And now it is an excellent epistemological exercise for the reader to once again examine each of the examples presented above in the light of these considerations. It will be clear to him that Aristotle's theory of formal abstraction offers a full explanation of the problem of the necessity of the immediate judgments of mathematics; the judgments are necessary although we need experience to affirm them, although we abstract these truths from sense-data. Hence Mathematics were simply called by Aristotle: "the product of abstraction".

In each of the examples we find in our mind an intuition of the external senses, an intuition of the imagination, but in addition, an intellectual intuition which considers what the phantasm represents, so that in it the necessary nexus may be intuited: "This is as it were to see with the intellect". Because it inspects the phantasm, intuition will be conjoined with abstraction; abstraction is present, but because it is formal, it is accompanied by intellectual intuition.

6. - NOTE 1. In formal abstraction a form is abstracted from some matter. We should note that here it is not always a question of a physical form related to a physical matter. Here the "form" is an element in the object which is clear to us; the matter is an element in the same object which is more or less obscure to us, as in obvious from our whole exposition. And so this matter from which we abstract, can be physically a form. Thus in all our examples we must abstract from colours, from the other qualities (except shape), from the specific nature of the bodies, which are all

physical forms or presuppose forms. But these physical forms are less clear to the human mind than quantity itself, whether discrete or continuous; and therefore they are like matter less knowable.

Let us hear S. Thomas on this point (S.Th. q.85 al ad 2): "Quantities such as numbers, and shapes, which are the limits of quantities, can be considered without sensible qualities, which means they are abstracted from sensible matter".

We should note further: those quantities are not abstractéd from the subject of which they are the forms, that is from the substance; for we are always considering quantified being. Thus immediately after the words quoted S.Thomas continues:

"However those quantities cannot be considered without the understanding of the substance which is determined by quantity, for that would be to abstract them from common intelligible matter." The matter, therefore, from which one abstracts is not always physical matter, just as the form which is abstracted is not always a physical form.

S. Thomas, in the words quoted, calls quantified substance "intelligible matter"; this confirms what we have said: in formal abstraction the form is that which is clear to us, for this 'intelligible matter' is that which remains in our mind as a result of formal abstraction. Aristotle uses the same expression 'intelligible matter' to describe extended being. It is called "matter" because from it are made mathematical figures (shapes); and 'intelligible' because it is entirely clear to the human mind.

It is called matter because it is the potency for those figures which are its acts. This Aristotelian notion of potency can and should be used to elucidate many questions which arise in the modern philosophy of mathematics.

7. - Note 2. This as much as we need to say on the problem of necessity. Again it must be noted that in what we have said we never considered points,

lines, straight lines, surfaces according to their strict definition, but only insofar as they can be perceived by the senses and which are roughly such.

With respect to these, all that we have said is valid. But for this reason we have not yet got the most general principles; but rather specific principles which are in fact necessary. We always directed our attention to a definite case in the phantasm (therefore numerically determined) but of this case (determined in the intellect not numerically, in particular, but specifically specie specialissima) the necessary judgment was already verified. Some cases also had straightaway a greater universality but not yet a perfect universality; for this the problem of exactitude must also be considered. For: when we were speaking of the divisibility of a line we did not yet discuss divisibility to infinity, for to establish this points strictly so called must be considered. So in the axiom of Pasch, the line through the vertex of a triangle must not come too close to one of the sides, for then the axiom must be considered which says that two straight lines cannot have two common points, and this again supposes the problem of exactitude. But all this has no influence on the problem of necessity, which is thus solved independantly of this universality.

We will note one further point. From what we have said it is clear that those moderns are wrong who want to totally equiperate geometry (applied to natural bodies) to the physical sciences. They do so because of the difficulties arising from the problem of exactitude. Now, even if this problem were insoluble, this would still be true: the approximative truths of geometry, with which we were dealing here, are entirely different from physical laws; this because of the intuition of necessity which is present in the former but absent in the latter. This also ia an experimental fact, a "finding" but an intellectual one. And this intuition must be put down to the formal abstraction of which the human mind is capable in the area of mathematics.

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